MAT 513 Homework 1

Due Wednesday, February 1

1. Show that in a field $\mathbb{F}$ we cannot divide by zero. In other words, show that there exists no element $x \in \mathbb{F}$ with the property that $x \cdot 0 = 1$.

Hint: Show that for every $y \in \mathbb{F}$ we have $y \cdot 0 = 0$. Then argue by contradiction and use the fact that $0 \neq 1$.

2. We saw in class that the set $\mathbb{Z}_2 = \{0, 1\}$ is a field when $+$ and $\cdot$ are computed modulo[†] 2, and we noted that something analogous is true for a set $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ with any prime p (i.e., $p = 2, 3, 5, 7, 11, \ldots$). Show that the set $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ is *not* a field when arithmetic is done modulo 4.

Hint: Show that in a field $\mathbb{F}$, if $a \cdot b = 0$, then $a = 0$ or $b = 0$. Is this true for the set $\mathbb{Z}_4$?

3. Show that $\sqrt{3}$ is irrational.

Hint: Argue by contradiction. Use the fact that if p is a natural number such that 3 divides p^2, then 3 also divides p. Then justify this fact using the *prime factorization theorem*.

4. Show that for every $a, b, c \in \mathbb{R}$ we have

- **(a)** $|a+b| \leq |a| + |b|$,
- **(b)** $||a| - |b|| \leq |a - b|$, and
- **(c)** $|a - c| \leq |a - b| + |b - c|$.

Then explain why the first (or third) inequality is called the "triangle inequality".

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function.

- **(a)** Given sets $A, B \subset \mathbb{R}$, what is the relation between $f(A \cup B)$ and $f(A) \cup f(B)$? Particularly, is one set contained in the other and are the sets equal to each other? Similarly, what is the relation between $f(A \cap B)$ and $f(A) \cap f(B)$? Prove your claims.
- **(b)** If you conclude that the two sets are not equal to each other in general, then provide an example of a function f and sets A, B illustrating your conclusion.
- **(c)** For a set $E \subset \mathbb{R}$, define $f^{-1}(E) = \{x \in \mathbb{R} : f(x) \in E\}$ [‡]. What is the answer to part (a) if $f(A), f(B), f(A \cup B)$, and $f(A \cap B)$ are replaced with $f^{-1}(A), f^{-1}(B), f^{-1}(A \cup B)$, and $f^{-1}(A \cap B)$, respectively?

6. Use induction to prove that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

Hint: Observe that $5^{2n+2} - 1 = 5^{2n+2} - 5^2 + 5^2 - 1$.

7. Study but do not submit Exercises 1.2.8 and 1.2.11.

[†] That is, we take the result to be the remainder after we divide it 2.

[‡] The preimage $f^{-1}(E)$ of a set E makes sense for every function f, even if the inverse function f^{-1} does not exist.