Homework 7 (due 4/4)

MAT 342: Applied Complex Analysis

Read Sections 48–50 and 52–53 from Chapter 4.

Problem 1. Show that $\int_C f(z)dz = 0$, where C is the unit circle, positively oriented, and

(i) $f(z) = \text{Log}(iz+3)$ (ii) $f(z) = \frac{1}{z^4+5i}$

Problem 2. Let C be the circle of radius 100, centered at the origin and positively oriented. The goal of this problem is to compute

$$\int_C \frac{1}{z^2-3z+2}dz.$$

(i) Decompose $\frac{1}{z^2-3z+2}$ into its partial fractions.

(ii) Compute $\int_{C_1} \frac{1}{z-1}dz$ and $\int_{C_2} \frac{1}{z-2}dz$, where C_1 is the circle of radius $1/4$, centered at 1 and positively oriented, and C_2 is the circle of radius $1/4$, centered at 2 and positively oriented.

(iii) Use the Theorem of Section 53 (Cauchy-Goursat Theorem for multiply connected domains) or its Corollary to evaluate $\int_C \frac{1}{z^2-3z+2}dz$. Explain carefully why the Theorem or the Corollary can be applied.

Problem 3. Let C_1 be the positively oriented square with vertices at the points $1+i$, $-1+i$, $-1-i$, $1-i$, and let C_2 be the positively oriented circle of radius 4, centered at the origin. Show that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz,$$

when

(i) $f(z) = \frac{e^z}{z^2 \sin(z/2)}$ (ii) $f(z) = \frac{1}{10z^3+3i}$

Problem 4.

(i) Let C_0 be the positively oriented circle of radius $R > 0$, centered at the point $1+i$. Compute

$$\int_{C_0} (z-1-i)^n dz,$$

where $n \in \mathbb{Z}$. Note that the answer depends on n.

(ii) Let C be any positively oriented simple closed contour surrounding the point $1+i$. Compute

$$\int_{C} (z-1-i)^n dz,$$

where $n \in \mathbb{Z}$. Again, the answer depends on n.

Problem 5. Let C denote the positively oriented boundary of the half disk $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$, and let $f(z)$ be defined by

$$f(z) = \sqrt{r}e^{i\theta/2}, \text{ where } z = re^{i\theta},\ r > 0,\ -\pi/2 < \theta < 3\pi/2.$$

We also define $f(0) = 0$. Note that f is continuous on the contour C, hence $\int_C f(z)dz$ is defined. Compute $\int_C f(z)dz$ by parametrizing the contour C (note that C consists of a semi-circle and two radii). Explain why we cannot apply the Cauchy-Goursat Theorem in this case.