Homework 4 (due 2/28)

MAT 342: Applied Complex Analysis

Read Sections 26–29 from Chapter 2 and Section 30 from Chapter 3.

Problems from the textbook:
§26: 1(b)(d), 2, 4, 7
§27: 1
§29: 2,4

Additional problems to hand in:

Problem 1. Consider the function $g(z) = \sqrt{r}e^{i\theta/2}$, where $z = re^{i\theta}$, $r > 0$, $-\pi < \theta < \pi$. This function is a branch of the square root and is analytic in its domain of definition with $g'(z) = \frac{1}{2g(z)}$, as we proved in the lecture.

(i) Sketch the domain of definition of g.

(ii) Find the image of the open set $S = \{z : \mathrm{Re}(z) > 1\}$ under the function $h(z) = 2z - 2 + i$.

(iii) Explain why the composition $g \circ h(z) = g(h(z))$ is defined in the open set S.

(iv) Calculate the derivative of $g \circ h$, using the chain rule.

Problem 2. Consider the function $g(z) = \ln r + i\theta$, where $z = re^{i\theta}$, $r > 0$, $-\pi < \theta < \pi$. Using the polar expression of the Cauchy-Riemann equations, show that the function g is analytic in its domain of definition, and show that $g'(z) = 1/z$. The function g is called *a branch of the complex logarithm*.

(i) Find the image of the first quadrant $S = \{z = x + iy : x > 0,\ y > 0\}$ under the map $h(z) = z^2 + 1$.

(ii) Explain why the composition $g \circ h(z) = g(h(z))$ is defined in the open set S.

(iii) Calculate the derivative of $g \circ h$.

Problem 3. Consider the unit circle $S = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : 0 \le \theta < 2\pi\}$ and write $z = re^{i\theta}$.

(i) What is the image of the circle S under the function $f(z) = \theta + i\cos\theta = (\theta, \cos\theta)$?

(ii) What is the image of the circle S under the function $g(z) = \cos\theta + i0 = (\cos\theta, 0)$?

(iii) What is the image of the circle S under the function $h(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$?