Homework 11 (due 5/2)

MAT 342: Applied Complex Analysis

Read Sections 74–82 from Chapter 6.

Problem 1. Explain why 0 is an isolated singular point and find the residue at $z = 0$ of each of the following functions:

(i) $z \cos(1/z)$ (ii) $\frac{\sinh z}{z^4(1-z^2)}$ (iii) $\frac{1}{z^4 \sin z}$

Problem 2. Evaluate the integral of each of the following functions around the positively oriented circle $|z| = 4$:

(i) $z^2 e^{1/z}$ (ii) $\frac{z+1}{(z-1)(z-5)z}$

Problem 3. Evaluate the integrals:

(i)
$$\int_C \frac{5z-8}{(z-1)(z-2)} dz,$$
where C is the positively oriented circle $|z| = 3$ using the *residue at infinity* of the function $\frac{5z-8}{(z-1)(z-2)}$.

(ii)
$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)} dz,$$
where C is the positively oriented circle $|z-2| = 2$.

Problem 4. Find the isolated singular points of the following functions and determine whether they are removable singular points, essential singular points, or poles:

(i) $ze^{1/z}$ (ii) $\frac{\sin z}{z}$ (iii) $\frac{\cos z}{z(z-\pi/2)}$ (iv) $\left(\frac{z}{2z+3i}\right)^5$

Problem 5. Show that the singular points of the following functions are poles. Determine their order and find the corresponding residues.

(i) $\frac{1-\cos z}{z^3}$ (ii) $\frac{e^{-3z}-1}{z^5(z-1)}$

Problem 6. Let f be a function that is analytic in a disk $|z - z_0| < R$. Note that f has a Taylor expansion in $|z - z_0| < R$.

(i) The function $g(z) = \frac{f(z)}{z-z_0}$ is analytic in the annulus $0 < |z - z_0| < R$. Write the Laurent expansion of g in the annulus $0 < |z - z_0| < R$, using the Taylor expansion of f.

(ii) Show that if $f(z_0) = 0$, then the point z_0 is a removable singular point for the function g.

(iii) Show that if $f(z_0) \neq 0$, then the point z_0 is a simple pole of the function g. In this case, find the residue of g at $z = z_0$.