Dr. Simon Albrecht *Spring 2016*

Name: ______________________ **ID:** ______________

MAT 342 Applied Complex Analysis
Final Exam Example

May 2016

1. (12 pts, 4 pts each)

   a) Define the notion *complex differentiable.*

   Let $S \subset \mathbb{C}$ be an open set and let $f : S \to \mathbb{C}$ be a function. Let $z_0 \in S$. The function f is called (complex) differentiable at z_0 if the limit

   $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

   exists.

   b) Define the *principle branch of the logarithm.*

   The principle branch of the logarithm is defined by

   $$\mathrm{Log}(z) = \ln(|z|) + i\mathrm{Arg}(z),$$

   where $z \in \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \leq 0\}$ and $-\pi < \mathrm{Arg}(z) < \pi$.

   c) State *Cauchy's residue theorem.*

   Let C be a simple closed, positively oriented contour, and let f be a function which is analytic on C and inside C with the possible exception of finitely many points z_k $(k = 1, \ldots, n)$ inside C. Then

   $$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_0} f(z).$$

Continue on page 2

Name: ________________________________ **ID:** __________________

2. (12 pts, 4 pts each)

a) Find the multiplicative inverse of $3+4i$ and write the solution in rectangular form.

b) Find all $z \in \mathbb{C}$ such that $z^2 = 4i$.

c) Prove the triangle inequality: For all $z, w \in \mathbb{C}$, the inequality

$$|z+w| \leq |z| + |w|$$

holds.

a)

$$(3+4i)^{-1} = \frac{1}{3+4i} = \frac{3-4i}{9+16} = \frac{3}{25} - i\frac{4}{25}.$$

b) We have $4i = 4e^{i\frac{\pi}{2}}$. Thus, the two complex roots are

$$\sqrt{4}e^{i\frac{\pi}{4}} = 2e^{i\frac{\pi}{4}} = 2\frac{1}{\sqrt{2}} + i2\frac{1}{\sqrt{2}} = \sqrt{2} + i\sqrt{2}$$

and

$$\sqrt{4}e^{i\left(\frac{\pi}{4}+\pi\right)} = -2e^{i\frac{\pi}{4}} = -\sqrt{2} - i\sqrt{2}.$$

c) Let $z, w \in \mathbb{C}$. Since $|z|^2 = z\overline{z}$, we get

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + z\overline{w} + \overline{z\overline{w}} + |w|^2 = |z|^2 + 2\mathrm{Re}(z\overline{w}) + |w|^2 \\ &\leq |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2. \end{aligned}$$

Thus,

$$|z+w| \leq |z| + |w|.$$

Continue on page 3

Name: ______________________ **ID:** ______________

3. (10 pts) Find all $z \in \mathbb{C}$ such that

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Proof. We have for $z \neq 1$

$$z^4 + z^3 + z^2 + z + 1 = \frac{z^5 - 1}{z - 1}$$

(partial sum of the geometric series). Thus,

$$z^4 + z^3 + z^2 + z + 1 = 0 \Leftrightarrow (z^5 = 1 \text{ and } z \neq 1).$$

Hence, all solutions of the equation are the non-trivial 5^{th} roots of unity, i.e.

$$e^{i\frac{2\pi}{5}} \quad , \quad e^{i\frac{4\pi}{5}} \quad , \quad e^{i\frac{6\pi}{5}} \quad , \quad e^{i\frac{8\pi}{5}}.$$

□

Continue on page 4

Name: ______________________ **ID:** ______________

4. (12 pts) Let f be an entire function such that

$$f(z) = f(z+1) = f(z+i)$$

for all $z \in \mathbb{C}$. Prove that f is constant.

Proof. Let $Q = \{z \in \mathbb{C} \mid 0 \leq \text{Re}z, \text{Im}z \leq 1\}$. Then for any $w \in \mathbb{C}$ there exists some $z \in Q$ such that $f(z) = f(w)$ (write $w = a + ib = (n+s) + i(m+r)$ for some $n, m \in \mathbb{Z}$ and $0 \leq s, r < 1$). Since Q is bounded and closed (i.e. compact) and f is continuous on Q, f is bounded on Q. Due to the argument above, f is bounded on all of $\mathbb{C}$. Thus, f is a bounded entire function which must be constant by Liouville's theorem. □

Continue on page 5

Name: ______________________________ **ID:** ________________

5. (10 pts) Let p be a polynomial of degree d_p and let q be a polynomial of degree d_q with $\max\{d_p, d_q\} \geq 1$. Assume that q is not constantly 0 and that p and q do not share a common zero. Let $f : \mathbb{C} \setminus \{z \in \mathbb{C} \mid q(z) = 0\} \to \mathbb{C}$ be given by

$$f(z) = \frac{p(z)}{q(z)}.$$

Let $z_0 \in \mathbb{C}$. Prove that there exists some $z \in \mathbb{C}$ such that $f(z) = z_0$.

Proof. We have

$$f(z) = z_0 \Leftrightarrow \frac{p(z)}{q(z)} = z_0 \Leftrightarrow p(z) = z_0 q(z) \Leftrightarrow p(z) - z_0 q(z) = 0.$$

Since p and q do not share a common zero, the zeros of q can't be solutions. But $p - z_0 q$ is a polynomial of degree $\max\{d_p, d_q\} \geq 1$. Hence, it has at least one zero $z \in \mathbb{C}$ by the Fundamental Theorem of Algebra. For this zero, $f(z) = z_0$ holds. $\square$

Continue on page 6

Name: ______________________________ **ID:** ________________

6. (12 pts) Find the Laurent series of

$$f(z) = \frac{1}{(z-1)(z-3)}$$

in $\{z \in \mathbb{C} \mid 0 < |z-1| < 2\}$.

Proof. We have for $z \neq 1$ and $z \neq 3$

$$\frac{-1}{2(z-1)} + \frac{1}{2(z-3)} = \frac{-(z-3)+(z-1)}{2(z-1)(z-3)} = f(z).$$

For $z \in \mathbb{C}$ with $0 < |z-1| < 2$, we have $\frac{|z-1|}{2} < 1$ and thus

$$\begin{aligned} f(z) &= \frac{-1}{2(z-1)} + \frac{1}{2(z-3)} = \frac{-1}{2(z-1)} + \frac{1}{2((z-1)-2)} \\ &= \frac{-1}{2(z-1)} + \frac{1}{4}\frac{1}{\frac{z-1}{2}-1} = \frac{-1}{2(z-1)} - \frac{1}{4}\frac{1}{1-\frac{z-1}{2}} \\ &= \frac{-1}{2(z-1)} - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \frac{-1}{2(z-1)} - \sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}. \end{aligned}$$

□

Continue on page 7

Name: ____________________________ **ID:** ________________

7. (12 pts, 4 pts each) Let
$$f(z) = \frac{1}{(z-2)(z-4)}.$$
Find the contour integrals of f along the circles about the origin of radius 1, 3 and 5, taken in counterclockwise direction.

Proof. Define curves $\gamma_1, \gamma_3, \gamma_5 : [0, 2\pi] \to \mathbb{C}$ by $\gamma_1(t) = e^{it}$, $\gamma_3(t) = 3e^{it}$, $\gamma_5(t) = 5e^{it}$. These curve parametrise the circles about the origin of radius 1, 3 and 5, all in counterclockwise direction.

As a rational function, f is analytic in the whole plane with the only exceptions being the zeros of the denominator, i.e. f is analytic in $\mathbb{C} \setminus \{2, 4\}$. In particular, f is analytic inside and on γ_1. By the Cauchy-Goursat theorem, this yields
$$\int_{\gamma_1} f(z)dz = 0.$$
Furthermore, we have that 2 lies inside γ_3, but 4 lies outside γ_3. By Cauchy's residue theorem, this yields
$$\int_{\gamma_3} f(z)dz = 2\pi i \operatorname*{Res}_{z=2} f(z)$$
and since both 2 and 4 lie inside γ_5
$$\int_{\gamma_5} f(z)dz = 2\pi i \left(\operatorname*{Res}_{z=2} f(z) + \operatorname*{Res}_{z=4} f(z) \right).$$
Since both 2 and 4 are simple poles of f ($\frac{1}{z-4}$ and $\frac{1}{z-2}$ are analytic and nonzero at 2 and 4, respectively), we get
$$\operatorname*{Res}_{z=2} f(z) = \frac{1}{2-4} = -\frac{1}{2} \quad \text{and} \quad \operatorname*{Res}_{z=4} f(z) = \frac{1}{4-2} = \frac{1}{2}.$$
Thus,
$$\int_{\gamma_3} f(z)dz = -\pi i \quad \text{and} \quad \int_{\gamma_5} f(z)dz = 0.$$
□

Continue on page 8

Name: ____________________ **ID:** __________

8. (20 pts, 10 pts each) Compute both

   a)
$$\int_0^\infty \frac{1}{1+x^4}dx \quad \text{and}$$

   b)
$$\int_{-\infty}^\infty \frac{x\sin(ax)}{x^4+4}dx \quad \text{where } a>0$$

using residues.

Proof. For $R>0$, we define $\gamma_1 : [-R,R] \to \mathbb{C}$, $\gamma_1(t)=t$, and $\gamma_2 : [0,\pi] \to \mathbb{C}$, $\gamma_2(t) = Re^{it}$. Furthermore, let $\gamma_R = \gamma_1+\gamma_2$. This curve consists of the real integral $[-R,R]$ and the semicircle of radius R in the upper half plane, taken in positive orientation.

a) The integrand $f(z) = \frac{1}{z^4+1}$ is analytic in the entire plane with the only exception being its singular points which are the 4^{th} roots of -1, i.e.

$$e^{i\frac{\pi}{4}} \quad , \quad e^{i\frac{3\pi}{4}} \quad , \quad e^{i\frac{5\pi}{4}} \quad , \quad e^{i\frac{7\pi}{4}}.$$

Only $e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$ and $e^{i\frac{3\pi}{4}} = \frac{-1+i}{\sqrt{2}}$ lie in the upper half plane, $e^{i\frac{5\pi}{4}} = \frac{-1-i}{\sqrt{2}}$ and $e^{i\frac{7\pi}{4}} = \frac{1-i}{\sqrt{2}}$ both lie in the lower half plane. There is no singular point on the real line. If we choose $R>1$, then both $e^{i\frac{\pi}{4}}$ and $e^{i\frac{3\pi}{4}}$ lie inside γ_R.

Both points are simple poles of f ($f=p/q$, $p(z)=1$, $q(z)=z^4+1$, $q'(z)=4z^3$, $q\left(e^{i\frac{\pi}{4}}\right) = 0 = q\left(e^{i\frac{3\pi}{4}}\right)$, $q'\left(e^{i\frac{\pi}{4}}\right) \neq 0 \neq q'\left(e^{i\frac{3\pi}{4}}\right)$). Thus,

$$\operatorname*{Res}_{z=e^{i\frac{\pi}{4}}} f(z) = \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3} = \frac{1}{4e^{i\frac{3\pi}{4}}} = \frac{-1}{4}e^{i\frac{\pi}{4}} \quad \text{and} \quad \operatorname*{Res}_{z=e^{i\frac{3\pi}{4}}} = \frac{1}{4e^{i\frac{9\pi}{4}}} = \frac{-1}{4}e^{i\frac{3\pi}{4}}.$$

Using Cauchy's residue theorem, we get

$$\int_{\gamma_R} f(z)dz = 2\pi i\left(-\frac{1}{4}e^{i\frac{\pi}{4}} - \frac{1}{4}e^{i\frac{3\pi}{4}}\right) = \frac{-\pi i}{2\sqrt{2}}(1+i+(-1+i)) = \frac{\pi}{\sqrt{2}}.$$

Furthermore, $|f(z)| \leq \frac{1}{R^4-1}$ for z on γ_2 and $L(\gamma_2) = \pi R$. Thus,

$$\left|\int_{\gamma_2} f(z)dz\right| \leq \frac{\pi R}{R^4-1} \to 0 \quad \text{as} \quad R\to\infty.$$

Thus,

$$\frac{\pi}{\sqrt{2}} = \lim_{R\to\infty}\int_{\gamma_R} f(z)dz = \lim_{R\to\infty}\left(\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz\right)$$
$$= \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_{\gamma_2} f(z)dz = P.V.\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx + 0.$$

Since $\frac{1}{x^4+1}$ is an even function, we get

$$P.V.\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx = \int_{-\infty}^{\infty}\frac{1}{x^4+1}dx = 2\int_{0}^{\infty}\frac{1}{x^4+1}dx$$

and hence

$$\int_{0}^{\infty}\frac{1}{x^4+1}dx = \frac{\pi}{2\sqrt{2}}.$$

b) The function $f(z) = \frac{z}{z^4+4}$ has four singular points at the 4^{th} roots of -4, i.e. at

$$\sqrt{2}e^{i\frac{\pi}{4}} \quad , \quad \sqrt{2}e^{i\frac{3\pi}{4}} \quad , \quad \sqrt{2}e^{i\frac{5\pi}{4}} \quad , \quad \sqrt{2}e^{i\frac{7\pi}{4}}.$$

As in a), only the first two lie in the upper half plane, the other two in the lower half plane, and none on the real line. Write $z_1 = \sqrt{2}e^{i\frac{\pi}{4}}$ and $z_2 = \sqrt{2}e^{i\frac{3\pi}{4}}$. For $R > \sqrt{2}$, both z_1 and z_2 also lie inside γ_R. Since $f(z)e^{iaz}$ has the same singular points as $f(z)$, we get by Cauchy's residue theorem

$$\int_{\gamma_R} f(z)e^{iz}dz = 2\pi i\left(\operatorname*{Res}_{z=z_1} f(z)e^{iaz} + \operatorname*{Res}_{z=z_2} f(z)e^{iaz}\right).$$

With the same argument as in a), we see that z_1 and z_2 are simple poles of $f(z)e^{iaz}$ and the residues are

$$\operatorname*{Res}_{z=z_1} f(z)e^{iaz} = \frac{z_1 e^{iaz_1}}{4z_1^3} = \frac{e^{iaz_1}}{4z_1^2} = \frac{e^{iaz_1}}{8e^{i\frac{\pi}{2}}} = \frac{e^{iaz_1}}{8i}$$

and

$$\operatorname*{Res}_{z=z_2} f(z)e^{iaz} = \frac{z_2 e^{iaz_2}}{4z_2^3} = \frac{e^{iaz_2}}{4z_2^2} = \frac{e^{iaz_2}}{8e^{i\frac{3\pi}{2}}} = \frac{e^{iaz_2}}{-8i}.$$

Since $z_1 = \sqrt{2}e^{i\frac{\pi}{4}} = 1+i$ and $z_2 = -1+i$, we get

$$\int_{\gamma_R} f(z)e^{iaz}dz = \frac{2\pi i}{8i}\left(e^{iaz_1} - e^{iaz_2}\right) = \frac{\pi}{4}\left(e^{ia-a} - e^{-ia-a}\right)$$
$$= \frac{\pi e^{-a}}{4}\left(e^{ia} - e^{-ia}\right) = \frac{\pi e^{-a}2i}{4}\sin(a).$$

Name: ______________________ **ID:** ______________

Since

$$\frac{\pi e^{-a}2i}{4}\sin(a) = \int_{\gamma_R} f(z)e^{iaz}dz = \int_{\gamma_1} f(z)e^{iaz}dz + \int_{\gamma_2} f(z)e^{iaz}dz$$
$$= \int_{-R}^{R} \frac{x}{x^4+4}e^{iax}dx + \int_{\gamma_2} f(z)e^{iaz}dz,$$

we get

$$\frac{\pi e^{-a}}{2}\sin(a) = \mathrm{Im}\left(\int_{-R}^{R} \frac{x}{x^4+4}e^{iax}dx + \int_{\gamma_2} f(z)e^{iaz}dz\right)$$
$$= \int_{-R}^{R} \frac{x\sin(ax)}{x^4+4}dx + \mathrm{Im}\left(\int_{\gamma_2} f(z)e^{iaz}\right)$$

For $t \in [0,\pi]$, we have

$$f(\gamma_2(t))e^{ia\gamma_2(t)} = \frac{Re^{it}}{R^4e^{i4t}+4}e^{iaR(\cos(t)+i\sin(t))} = \frac{re^{it}}{R^4e^{i4t}+4}e^{iaR\cos(t)}e^{-aR\sin(t)}.$$

Since $\sin(t) \geq 0$ for these t and both $a, R > 0$, we get

$$\left|f(\gamma_2(t))e^{ia\gamma_2(t)}\right| \leq \frac{R}{R^4-4}e^{-aR\sin(t)} \leq \frac{R}{R^4-4}$$

which implies with $L(\gamma_2) = \pi R$

$$\left|\mathrm{Im}\left(\int_{\gamma_2} f(z)e^{iaz}dz\right)\right| \leq \left|\int_{\gamma_2} f(z)e^{iaz}dz\right| \leq \frac{\pi R^2}{R^4-4} \to 0 \quad \text{as} \quad R \to \infty.$$

Combined,

$$\frac{\pi e^{-a}}{2}\sin(a) = \lim_{R\to\infty}\int_{-R}^{R} \frac{x\sin(ax)}{x^4+4}dx = P.V. \int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}dx.$$

Since $\frac{x\sin(ax)}{x^4+4}$ is even, we get

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}dx = \frac{\pi e^{-a}}{2}\sin(a).$$

□