MAT 342 - Applied Complex Analysis
Final Exam, Fall 2017

Tuesday, December 19, 2:15 pm - 5:00 pm

Name : ___________________ Grades:

ID : ___________________

Instructions:

Complete all the problems below. There are 10 problems in total, each of which values 10 points. In order to receive full credit for any problem you must show all of your work, and must provide full justification for your answer.

All answers should be simplified (e.g. $\frac{2}{4}$ should be $\frac{1}{2}$). Take care not to miss any questions.

You may not use calculators or other electronic devices, including cell phones.

Be sure to write your name and student ID on each page you hand in.

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1. (2 pts each) Tell whether the following statements are true or false. If a statement is true, write "T" in the bracket; otherwise write "F". You do not need to give the proof.

( ) $\overline{(2-i)^2} = 3 - 4i$;

( ) The polar form (exponential form) of $z = -(\sqrt{3} + i)$ is $z = -2e^{i\frac{\pi}{6}}$;

( ) $|z - 1 + 3i| = 2$ represents the circle centered at $(1, -3)$ whose radius is 2;

( ) $f(z) = \sin z$ is not bounded for $z \in \mathbb{C}$;

( ) For all $z_1, z_2 \in \mathbb{C}$, $\log(z_1 z_2) = \log z_1 + \log z_2$.

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2. Consider the set $M_1 = \{z = re^{i\theta} \in \mathbb{C} \mid 0 \le r \le \sqrt{2},\ 0 \le \theta \le \frac{\pi}{4}\}$.

   (a) (5 pts) Find and sketch the image set M_2 of M_1 under the map $w = z^2$.

   (b) (5 pts) Find and sketch the image set M_3 of M_2 under the map $\xi = e^w$.

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3. (a) (5 pts) Find the points where $f(z) = z\mathrm{Im}z$ is differentiable and compute $f'(z)$ at those points.

   (b) (5 pts) Suppose $g(z) = u + iv$ is entire. If $u(x, y) = e^x \sin y$, what equations does v satisfy? Solve them to get $v = v(x, y)$.

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4. (5 pts each) Calculate the following integrals.

   (a)
   $$\int_C (e^z + 1)\ dz$$
   where C is any path that starts from 0 and ends at $1 + \pi i$.

   (b)
   $$\int_C \overline{z}\ dz$$
   where C is the square with vertices $1 \pm i, -1 \pm i$ (counterclockwise oriented).

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5. (a) (4 pts) Let f be analytic everywhere inside and on a (positively oriented) simple closed contour C, and z_0 be a point interior to C. Write down Cauchy integral formulae for $f(z_0)$ and $f'(z_0)$;

   (b) (3 pts each) Suppose C is the circle $|z| = 1$. Use (a) to evaluate the integrals:

   1) $\displaystyle\int_C \frac{\sin z}{z(z^2+8)}dz$ 2) $\displaystyle\int_C \frac{\tan z}{(z-\frac{\pi}{4})^2}dz$

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6. (5 pts each)

   (a) Find the Taylor series expansion of $\frac{1}{z-2}$ **centered at** 1 and state the radius of convergence.

   (b) By differentiating (a), find the Taylor series expansion of $\frac{z-1}{(z-2)^2}$ **centered at** 1 and state the radius of convergence.

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7. (5 pts each) Find the Laurent series **centered at** 1 of the function

$$f(z) = \frac{z}{(z-1)(z-2)}$$

in the following regions:

(a) $0 < |z-1| < 1$.

(b) $|z-1| > 1$.

What is the residue of $f(z)$ at 1?

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8. (5 pts each) Calculate the following countour integrals using residues:

(a)

$$\int_{|z|=2} \tan z \; dz;$$

(b)

$$\int_{|z|=666} \frac{z^3 + 2z}{(z-i)^3}.$$

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9. (5 pts each) In each of the following, determine location of the isolated singular points, use Laurent series expansion to classify the isolated singular points, and calculate their residues.

   (a) $z^2 \exp\left(\frac{1}{z}\right)$;

   (b) $\frac{\cos z}{z}$.

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10. Let $f(z) = \frac{z+1}{z^2+4z+5}$. For $R > \sqrt{5}$, consider the semicircle $C_R : z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$.

(a) (3 pts) Calculate the integral

$$\int_{-R}^{R} \frac{(x+1)e^{ix}}{x^2+4x+5}\,dx + \int_{C_R} f(z)e^{iz}\,dz.$$

(b) (3 pts) Show that

$$\lim_{R\to\infty} \left| \int_{C_R} f(z)e^{iz}\,dz \right| = 0.$$

(c) (4 pts) Use (a)(b) to calculate the Cauchy principal values of the integrals

$$\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\,dx, \quad \int_{-\infty}^{\infty} \frac{(x+1)\sin x}{x^2+4x+5}\,dx.$$