MAT 341 – Applied Real Analysis
SPRING 2015

Midterm 2 – April 16, 2015

NAME: ______________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
TOTAL	

Problem 1: Consider the heat equation

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}$$

on the interval $0 < x < 2$, with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 10, \quad u(2,t) = 100, \qquad \text{for all } t > 0.$$

a) (8 points) What is the *steady-state* temperature distribution?

b) (12 points) Find all the product solutions $w(x,t) = \phi_n(x)T_n(t)$ that satisfy the PDE and the boundary conditions for the *transient* solution. You are **NOT** asked to find the general solution!

Problem 2: (20 points) Find the Fourier integral representation of the function $f(x)$ given below:

$$f(x) = \begin{cases} \pi & \text{if} \quad 0 < x < 1 \\ 0 & \quad\text{otherwise.} \end{cases}$$

Problem 3: (20 points) Consider the heat conduction problem in a metal rod of semi-infinite length that is insulated on the sides:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$u(0, t) = 0, \qquad t > 0,$$

whose initial temperature distribution is $u(x, 0) = f(x)$ for $0 < x < \infty$, where

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \quad \text{otherwise.} \end{cases}$$

Find the temperature $u(x, t)$ if we further assume that $u(x, t)$ remains finite as $x \to \infty$.

Problem 4:

a) (10 points) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the problem:

$$\phi'' + \lambda^2 \phi = 0, \quad 0 < x < 1$$
$$\phi(0) = 0, \quad \phi'(1) = 0$$

b) (10 points) Find the expression of the function $f(x) = x$, $0 < x < 1$ in terms of these eigenfunctions. Does this series converge at $x = 1$?

Problem 5: (20 points) Solve the vibrating string problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0;$$
$$u(0,t) = 0, \qquad u(1,t) = 0, \quad t > 0;$$
$$u(x,0) = \sin(3\pi x), \qquad 0 < x < 1;$$
$$\frac{\partial u}{\partial t}(x,0) = \sin(5\pi x), \qquad 0 < x < 1.$$

Explain why $u(x,t+1) = u(x,t)$, which means that the solution to this problem is a function that is periodic in time of period 1.

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

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