MAT 341 – Applied Real Analysis
Fall 2015

Midterm 2 – November 5, 2015

Solutions

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator. You **are** allowed to bring a note card to the exam (8.5 x 5.5in - front and back), but no other notes are allowed.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
TOTAL	

Problem 1: (12 points) The *telegraph equation* governs the flow of voltage, or current, in a transmission line and has the form:

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + ku = a^2\frac{\partial^2 u}{\partial x^2} + F(x,t), \qquad 0 < x < 100, \quad t > 0.$$

The coefficients c, k, a are constants related to electrical parameters in the line. Assuming that $F(x,t) = 0$ and $u(x,t) = \phi(x)T(t)$, carry out a separation of variables and find the eigenvalue problem for ϕ. Take the boundary conditions to be

$$\frac{\partial u}{\partial x}(0,t) = 0 \text{ and } u(100,t) = 0, \quad t > 0.$$

Find an ordinary differential equation that is satisfied by $T(t)$.

Solution. If we substitute $u(x,t) = \phi(x)T(t)$ we get $\phi T'' + c\phi T' + k\phi T = a^2\phi'' T$. Separation of variables gives

$$\frac{T'' + cT' + kT}{T} = a^2\frac{\phi''}{\phi} = \lambda, \qquad \text{where } \lambda \text{ is some real number.}$$

We get $a^2\phi'' - \lambda\phi = 0$ and $T'' + cT' + (k-\lambda)T = 0$, which is an ODE satisfied by T. The first boundary condition gives $\frac{\partial u}{\partial x}(0,t) = \phi'(0)T(t) = 0$ so $\phi'(0) = 0$. The second boundary condition gives $u(100,t) = \phi(100)T(t) = 0$, so $\phi(100) = 0$. $\square$

Problem 2: (20 points) Solve the heat problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(2,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < 2, \qquad \text{where} \quad f(x) = \begin{cases} T_0 & \text{if} \quad 0 < x < 1 \\ T_1 & \text{if} \quad 1 \le x < 2 \end{cases}$$

SOLUTION. We identify $a = 2$ and $k = 4$. The general solution to this equation is

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{2}\right) e^{-n^2\pi^2 t}.$$

The coefficients can be found from the initial condition $u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{2}\right) = f(x)$. We have $c_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{T_0+T_1}{2}$ and

$$\begin{aligned} c_n &= \int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx = \int_0^1 T_0 \cos\left(\frac{n\pi x}{2}\right)\,dx + \int_1^2 T_1 \cos\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{2T_0}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{2T_1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 \\ &= \frac{2(T_0 - T_1)}{n\pi}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

The solution is

$$u(x,t) = \frac{T_0+T_1}{2} + 2(T_0 - T_1)\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) e^{-n^2\pi^2 t}.$$

$\square$

Problem 3:

a) (12 points) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the problem:

$$\phi'' + \lambda^2\phi = 0, \quad 0 < x < 1$$

$$\phi(0) = 0, \qquad \phi'(1) - \phi(1) = 0$$

Is $\lambda = 0$ an eigenvalue?

Solution. If $\lambda = 0$ then $\phi'' = 0$ so $\phi(x) = Ax + B$. From $\phi(0) = 0$ we immediately find $B = 0$. However the relation $\phi'(1) - \phi(1) = 0$ does not give other information about A. We find $\phi(x) = Ax$ for $A \neq 0$. So $\lambda = 0$ is an eigenvalue.

If $\lambda \neq 0$ then $\phi(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$. The condition $\phi(0) = 0$ gives $C_1 = 0$. We can take $C_2 = 1$ at this step and write $\phi(x) = \sin(\lambda x)$. The condition $\phi'(1) - \phi(1) = 0$ gives $\lambda = \tan(\lambda)$. The eigenvalues are λ_n, the n^{th} root of the equation $\lambda = \tan(\lambda)$, for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are $\phi_n(x) = \sin(\lambda_n x)$. $\square$

(Problem 3 continued)

b) (5 points) Consider the function

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 < x < 0.5 \\ 1 - x & \text{if} \quad 0.5 \leq x < 1. \end{cases}$$

Suppose $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is the expansion of the function $f(x)$ in terms of the eigenfunctions $\phi_n(x)$ from part a). Write down a formula for the coefficients c_n. You are **not** asked to compute the coefficients.

SOLUTION. We have

$$c_n = \frac{\displaystyle\int_0^1 f(x)\phi_n(x)\,dx}{\displaystyle\int_0^1 \phi_n^2(x)\,dx}.$$

□

c) (7 points) To what value does the series converge at $x = 0.5$? What about at $x = 0$ and $x = 0.3$?

SOLUTION. The function has a jump discontinuity at $x = 0.5$ so the series converges to $\dfrac{f(.5-) + f(.5+)}{2} = \dfrac{1.5}{2} = \dfrac{3}{4}$. The function is continuous at $x = 0.3$ so the series converges to $f(0.3) = 0.6$. At $x = 0$, we have $\phi_n(0) = 0$ from the hypothesis so the series converges to 0. □

Problem 4: (22 points) Solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t > 0$$

$$u(x,t) \ \text{ bounded as } x \to \infty$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \quad \text{where} \quad f(x) = \begin{cases} \pi - x & \text{if } \ 0 < x < \pi \\ 0 & \text{if } \quad \pi \le x \end{cases}$$

Solution. The solution is given by

$$u(x,t) = \int_0^\infty A(\lambda)\cos(\lambda x)e^{-2\lambda^2 t}\,d\lambda,$$

where

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi}\int_0^\infty f(x)\cos(\lambda x)\,dx = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos(\lambda x)\,dx \\ &= 2\int_0^\pi \cos(\lambda x)\,dx - \frac{2}{\pi}\int_0^\pi x\cos(\lambda x)\,dx \\ &= \frac{2}{\lambda}\sin(\lambda x)\Big|_0^\pi - \frac{2}{\pi}\left(\frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}\right)\Big|_0^\pi \\ &= \frac{2\sin(\lambda\pi)}{\lambda} - \frac{2}{\pi}\frac{\cos(\lambda\pi)}{\lambda^2} - \frac{2\sin(\lambda\pi)}{\lambda} + \frac{2}{\pi\lambda^2} \\ &= \frac{2 - 2\cos(\lambda\pi)}{\pi\lambda^2}. \end{aligned}$$

Therefore the solution is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda^2}\cos(\lambda x)e^{-2\lambda^2 t}\,d\lambda.$$

□

Problem 5: (22 points) If an elastic string is *free* at one end, the boundary condition to be satisfied there is that $\dfrac{\partial u}{\partial x} = 0$. On the other hand, if it is *fixed* at one end, the boundary condition to be satisfied there is that $u = 0$. Find the displacement $u(x,t)$ in an elastic string of length $a = 1$, fixed at $x = 0$ and free at $x = a$, set in motion with no initial velocity from the initial position $u(x,0) = \sin\left(\dfrac{3\pi x}{2}\right)$.

a) State the boundary value problem that $u(x,t)$ satisfies. Include the initial conditions.

SOLUTION. The initial value-boundary value problem is the following:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0;\\
&u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) = 0, \qquad t > 0;\\
&u(x,0) = \sin\left(\frac{3\pi x}{2}\right), \qquad 0 < x < 1;\\
&\frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < 1.
\end{aligned}$$

$\square$

b) Find $u(x,t)$.

SOLUTION. We solve the associated eigenvalue problem and find $\lambda_n = \dfrac{(2n-1)\pi}{2}$, for $n = 1, 2, \ldots$. The general solution of this PDE is therefore

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\lambda_n ct)\sin(\lambda_n x) + b_n \sin(\lambda_n ct)\sin(\lambda_n x).$$

From $\dfrac{\partial u}{\partial t}(x,0) = 0$ we find that $b_n = 0$ for all n. From the initial condition $u(x,0) = \sin\left(\dfrac{3\pi x}{2}\right)$ we find that

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi x}{2}\right) = \sin\left(\frac{3\pi x}{2}\right)$$

The Fourier series is unique, so we just need to make the coefficients of the left-hand side equal to the coefficients of the right-hand side. This yields $a_2 = 1$ and $a_n = 0$ for all $n \neq 2$. The solution is then

$$u(x,t) = \cos\left(\frac{3\pi ct}{2}\right)\sin\left(\frac{3\pi x}{2}\right).$$

$\square$

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$