PRACTICE FINAL FOR MAT 341

In problems (1)-(4) below we consider a cylindrical rod centered along the x-axis in 3-dimensional space, from $x = 0$ to $x = a$; for each $0 \leq x \leq a$ the intersection of this rod with the plane containing $(x, 0, 0)$ and perpendicular to the x-axis is a disc D_x of area A. We assume that the physical properties of this rod are the same at each of its points: in particular its density function is a constant function ρ, and the heat capacity per unit mass for the rod is also a constant function c (see page 36). We assume that for each time $t \geq 0$ and for each $0 \leq x \leq a$ the temperature at each point of D_x is equal to the same value $u(x,t)$. Finally we assume that the cylindrical surface of the rod is insulated (the ends of the rod are not necessarily insulated).

(1) What does the *heat flux* function $q(x,t)$ measure? State *Fourier's law of heat conduction* for this rod.
Solution: See bottom of page 135 and top middle of page 137.

(2) Suppose that the rod is also insulated at its right hand end D_a and is kept at a constant temperature of 2 degrees celcius at its left hand end D_0.

(a) Give a mathematical description (some equations) of all these conditions placed on $u(x,t)$, $0 \leq x \leq a$ and $0 \leq t$.
Solution:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$u(0,t) = 2, \quad \frac{\partial u}{\partial x}(a,t) = 0$$

(b) Find a general solution to the equations in part (a).
Solution:

$$u(x,t) = 2 + \sum_{n=1}^{\infty} b_n sin(\lambda_n x) e^{-\lambda_n^2 kt}$$

where $\lambda_n = \frac{(2n-1)\pi}{2a}$.

(3) Let $H(x,t)$ denote the total heat contained within the portion of the rod between D_0 and D_x. Recall that $H(x,t) = \int_0^x \rho c A u(y,t) dy$ (see page 136 of text).

Suppose that the rod is insulated at its right hand end D_a and that $H(a,2) < H(a,0)$. Then show that $\frac{\partial u}{\partial x}(0,t_0) > 0$ holds for some $0 \leq t_0 \leq 2$.
Solution: The rod can loose heat only thru the disc D_0. It must loose heat thru the disc D_0 at some time $0 \leq t_o \leq 2$ because $H(a,2) < H(a,0)$. Thus the rate of heat flow thru D_0 at time t_o — which is equal to $q(0,t_o)A$ — must be negative; i.e. $q(0,t_o) < 0$. Since $q(x,t) = -\kappa \frac{\partial u}{\partial x}(x,t)$ (why?), it follows that $\frac{\partial u}{\partial x}(0,t_o) > 0$.

(4) Suppose that $u(x,t)$ satisfies $u(x,0) = 3$ in addition to the properties of problem (2)(a) above.

(a) Compute $H(a,0)$ and $limit_{t\to\infty} H(a,t)$.
Solution: $H(a,0) = 3a\rho cA$. $limit_{t\to\infty} H(a,t)$ should equal to the heat content of the bar for the steady state solution $v(x)$. Note that $v(x) = 2$; so the heat content for the steady state solution is $2a\rho cA$.

(b) Verify that $\frac{\partial u}{\partial x}(0,t) > 0$ for all $t > 0$. (**Hint:** Write $u(x,t)$ as an infinite series and compute its x-derivative term by term.)
Solution: $u(x,t)$ is the solution to the equations of (2)(a) and the initial condition $u(x,0) = 3$. Thus $u(x,t)$ is equal to the infinite series of given in (2)(b), where the b_n in (2)(a) are given by $b_n = \frac{2}{a}\int_0^a sin(\lambda_n x)dx = \frac{2}{a\lambda_n}$. Thus $\frac{\partial u}{\partial x}(0,t) = \sum_{n=1}^{\infty} \frac{2}{a} e^{-\lambda_n^2 kt}$, which is clearly positive for all $t > 0$.

(c) Use part (b) to verify that $H(a,t)$ is a decreasing function in t.
Solution: Using Fourier's Law, and part (b) of this problem, we conclude that $q(x,t) < 0$ for all t. Thus heat is flowing out of the rod at D_0 for all $t > 0$; implying that the heat content of the rod $H(a,t)$ is decreasing for all $t > 0$.

(5) For all $0 \le x \le \pi$ and $0 \le t$ suppose that the following equations hold for the function $u(x,t)$:

$$(i) \quad \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$$

$$(ii) \quad u(0,t) = 0, u(\pi,t) = 0$$

$$(iii) \quad u(x,0) = sin(x), \frac{\partial u}{\partial t}(x,0) = sin(x)$$

(a) find the d'Alembert solution to these equations.
Solution: $u(x,t) = \frac{sin(x+t)+sin(x-t)}{2} + \frac{cos(x-t)-cos(x+t)}{2}$

(b) find the Fourier type solution to these equations.
Solution: $u(x,t) = \sum_{n=1}^{\infty} sin(nx)(a_n cos(nt) + b_n sin(nt))$, where $a_n = \frac{2}{\pi}\int_0^{\pi} sin(x)sin(nx)dx$ and $b_n = \frac{2}{n\pi}\int_0^{\pi} sin(x)sin(nx)dx$. Thus

$$u(x,t) = sin(x)(cos(t) + sin(t)).$$

(c) does this vibrating string ever return to its original position?
Solution: Using the solution in (b) above, we see that $u(x,t)$ is periodic of period 2π in the t variable. So the string returns to its original position after 2π amount of time has elapsed.

(6) Show that if $u_1(x,t)$ and $u_2(x,t)$ both satisfy equations (i),(ii) in problem (5), then $u(x,t) = \alpha_1 u_1(x,t) + \alpha_2 u_2(x,t)$ also satisfies (i),(ii) in problem (5) for any real numbers α_1, α_2.
Solution: This uses the homogeneity of (i)(ii).

To verify (i):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sum_{i=1}^{2} \alpha_i (\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2}) = \sum_{i=1}^{2} \alpha_i \frac{1}{c^2} \frac{\partial^2 u_i}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.$$

To verifiy (ii): $u(0,t) = \sum_{i=1}^{2} \alpha_i u_i(0,t) = \sum_{i=1}^{2} \alpha_i 0 = 0$; $u(a,t) = \sum_{i=1}^{2} \alpha_i u_i(a,t) = \sum_{i=1}^{2} \alpha_i 0 = 0$.

(7) Do problem (11) on page 232 of the text.

(8) Suppose that $u(x,t)$, $0 \le x, t$, satisfies

$$\frac{\partial u}{\partial x}(x,t) = -\frac{1}{c} \frac{\partial u}{\partial t}(x,t)$$
$$u(0,t) = 0$$
$$u(x,0) = f(x)$$

for some given differentiable function $f(x)$.

(a) Show that $u(x,t)$ also satisfies

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}(x,t)$$

and

$$\frac{\partial u}{\partial t}(x,0) = -cf'(x).$$

Solution: The first equality is derived as follows:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial x}(\frac{-1}{c}\frac{\partial u}{\partial t}) = \frac{-1}{c}\frac{\partial}{\partial x}(\frac{\partial u}{\partial t}) = \frac{-1}{c}\frac{\partial}{\partial t}(\frac{\partial u}{\partial x}) = \frac{-1}{c}\frac{\partial}{\partial t}(\frac{-1}{c}\frac{\partial u}{\partial t}) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

The second equality is derived as follows:

$$\frac{\partial u}{\partial t}(x,0) = -c\frac{\partial u}{\partial x}(x,0) = -cf'(x).$$

(b) Solve for $u(x,t)$ in terms of the function f.

Solution: In part (a) we showed that u satisfies the wave equation and has initial position $u(x,0) = f(x)$ and initial velocity $\frac{\partial u}{\partial t}(x,0) = g(x) = -cf'(x)$. Thus — by d'Alembert — we have that

$$u(x,t) = \frac{1}{2}(f(x+ct) + f_o(x-ct) + G(x+ct) - G_e(x-ct))$$

where

$$G(x) = \frac{1}{c}\int_0^x g(s)ds = \frac{1}{c}\int_0^x -cf'(s)ds = -\int_0^x f'(s)ds = -(f(x) - f(0)).$$

Combining these last two equalities we get that

$$u(x,t) = \frac{1}{2}(f_o(x-ct) + f_e(x-ct)).$$

Note that $\frac{1}{2}(f_o + f_e) = \hat{f}$ — where $\hat{f}(x) = f(x)$ if $x \geq 0$ and $\hat{f}(x) = 0$ if $x < 0$.

(c) Give a physical description of the solution of part (b).
Solution: This is a traveling wave, moving from left to right.

(9) A real valued function $u(x, y)$ of the two real variables x, y is *harmonic* if it satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on its domain.

(a) If $u(x, y) = \sum_{0 \leq i+j \leq 3} a_{i,j} x^i y^j$, and u is harmonic in a disc of radius 2 centered at (-3,4), then prove that u is harmonic on the whole plane.
Solution: u is harmonic on the disc iff $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ holds on the disc.
Note that $\frac{\partial^2 u}{\partial x^2} = 2a_{2,1}y + 6a_{3,0}x$; and $\frac{\partial^2 u}{\partial y^2} = 2a_{1,2}x + 5a_{0,3}y$. Thus we have that u is harmonic on the disc iff

$$2a_{2,1} + 5a_{0,3} = 0$$

and

$$6a_{3,0} + 2a_{1,2} = 0.$$

Note that these last two equalities also equivalent to u being harmonic on the whole plane.

(b) It is a fact that if u is harmonic on a finite rectangle $\mathbb{R} = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then it takes on neither a maximum value nor a minimum value in the interior of this rectangle $\{(x, y) \mid a < x < b, c < y < d\}$. Prove this fact under the additional hypothesis that $\frac{\partial^2 u}{\partial x^2}$ does not vanish in the interior of the rectangle.
Solution: If u takes on a maximum or minimum at a point p inside of $\mathbb{R}$, then p must be a critical point for u. Now apply the second derivative test for u at p; you will see (using the hypothesis for u) that p is a saddle point for u.

(10) Consider the following 2-dimensional heat problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$u(x, 0, t) = 0, \quad u(x, b, t) = 0$$

$$u(0, y, t) = 3sin(\frac{2\pi}{b}y), \quad u(a, y, t) = -sin(\frac{5\pi}{b}y)$$

$$u(x, y, 0) = x + y$$

Find the steady state solution for this problem.

Solution: The steady state solution $v(x,y)$ satisfies the following equati0ns:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$v(x,0) = 0, v(x,b) = 0, v(0,y) = 3sin(\frac{2\pi}{b}y), v(a,y) = -sin(\frac{5\pi}{b}y).$$

So we may use the results of section 4.2 (with the roles of x, a and y, b reversed) to conclude that

$$v(x,y) = \sum_{n=1}^{\infty}(a_n e^{\lambda_n x} + b_n e^{\lambda_n x)})sin(\lambda_n y)$$

where $\lambda_n = \frac{n\pi}{b}$. We can solve for a_n, b_n by comparing the above form of $v(x,y)$ with the last two boundary conditions. Thus $3sin(\lambda_2 y) = v(0,y) = \sum_{n=1}^{\infty}(a_n + b_n)sin(\lambda_n y)$, which implies that

$$3 = a_2 + b_2$$

$$0 = a_n + b_n, \quad n \neq 2.$$

Also $-sin(\lambda_5 y) = v(a,y) = \sum_{n=1}^{\infty}(a_n e^{\lambda_n a} + b_n^{-\lambda_n a})sin(\lambda_n y)$, which implies

$$-1 = a_5 e^{\lambda_5 a} + b_5 e^{-\lambda_5 a}$$

$$0 = a_n e^{\lambda_n a} + b_n e^{-\lambda_n a}, \quad n \neq 5.$$

We can solve the preceeding 4 displayed equalities for a_n, b_n: in particular $a_n = 0 = b_n$ if $n \neq 2, 5$.