MAT 341 – Applied Real Analysis
Spring 2015

Final – May 18, 2015

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
6	
7	
TOTAL	

Problem 1: (12 points) Consider the function

$$f(x) = \begin{cases} 0 & \text{if} \quad -3 \le x < -1, \\ 1 & \text{if} \quad -1 \le x < 1, \\ 0 & \text{if} \quad 1 \le x < 3; \end{cases} \qquad\qquad f(x+6) = f(x).$$

a) Sketch the graph of f on the interval $[-7, 7]$.

b) Find the Fourier series for f. Explain why the series converges to 0.5 when $x = 7$.

Problem 2: (14 points) Suppose $u(x,t) = e^{-\lambda t} X(x)$ is a nontrivial solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}, \qquad 0 < x < a, \qquad t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad u(a,t) = 0, \quad t > 0.$$

Find an ordinary differential equation that is satisfied by $X(x)$. What initial conditions must $X(x)$ satisfy? Determine $X(x)$ and the possible values of λ.

Problem 3: (14 points) Consider the following eigenvalue problem

$$(e^{-x}\phi')' + e^{x}\gamma^2\phi = 0, \quad 0 < x < a$$
$$\phi(0) + \beta^2\phi'(0) = 0, \qquad \phi(a) + \beta^2\phi'(a) = 0$$

Check true or false, no other explanation is necessary.

a) This is a regular Sturm-Liouville problem for all values of the parameter β.

TRUE FALSE

b) If $\beta = 0$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then

$$\int_0^a \phi_2(x)\phi_4(x)\,dx = 0.$$

TRUE FALSE

c) If $\beta = 0$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then

$$\int_0^a \phi_2(x)\phi_4(x)e^{x}\,dx = 0.$$

TRUE FALSE

d) If $\beta = 0$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then

$$\int_0^a \phi_m(x)\phi_n(x)e^{x}\,dx = 0.$$

TRUE FALSE

e) If $\beta = 4$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then

$$\int_0^a \phi_3(x)\phi_5(x)e^{x}\,dx = 0.$$

TRUE FALSE

f) $\gamma = 0$ is not an eigenvalue, regardless of the parameter β.

TRUE FALSE

g) If $\beta = 0$ and $\phi_1, \phi_2, \phi_3, \ldots$ are eigenfunctions of this problem then $\sum_{n=1}^{\infty} c_n\phi_n(x) = e^{-x}$, for $0 < x < a$, where

$$c_n = \frac{\displaystyle\int_0^a \phi_n(x)\,dx}{\displaystyle\int_0^a \phi_n^2(x)e^{x}\,dx}.$$

TRUE FALSE

Problem 4: (20 points) Consider the *dispersive wave equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \gamma^2 u, \qquad 0 < x < a, \quad t > 0$$

subject to the following boundary conditions and initial conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(a,t) = 0, \qquad t > 0; \qquad\qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < a.$$

a) Set $u(x,t) = \phi(x)T(t)$ to separate the variables and find the associated equations for ϕ and T. Solve these equations and show that the solution $u(x,t)$ can be written as

$$u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(t\sqrt{\frac{n^2\pi^2}{a^2} + \gamma^2}\right) \cos\left(\frac{n\pi}{a}x\right).$$

(Problem 4 continued)

b) Find the formula for the coefficients c_n using $f(x)$.

c) By using trigonometric identities, rewrite the solution as

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} c_n \left[\cos\left(\frac{n\pi}{a}(x - b_n t) \right) + \cos\left(\frac{n\pi}{a}(x + b_n t) \right) \right]$$

Determine b_n, the speed of wave propagation. When is b_n independent of n?

Problem 5: (10 points) Consider the potential equation in a vertical strip $0 < x < a$, $2 < y$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \qquad 2 < y;$$
$$u(0, y) = 0, \qquad u(a, y) = 0, \qquad 2 < y;$$

We know that the *bounded* solutions to this equation are given by

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \exp\left(-\frac{n\pi y}{a}\right).$$

Find the coefficients c_n if in addition $u(x, 2) = 1$, $0 < x < a$. Is the solution periodic in y? What happens to $u(x, y)$ and $\frac{\partial u}{\partial x}(x, y)$ as $y \to \infty$?

Problem 6: (12 points) We know that the following functions $v(r,\theta)$:

$$1, \quad r^n\cos(n\theta), \quad r^{-n}\cos(n\theta), \quad r^n\sin(n\theta), \quad r^{-n}\sin(n\theta), \qquad (\text{where } n = 1, 2, \ldots)$$

are all solutions to the Laplace equation in polar coordinates: $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$. Let v_1 and v_2 be any two functions from the given list.

a) Any combination $c_1v_1 + c_2v_2$ is a solution of the Laplace equation.

Check true or false, no other explanation is necessary: $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

b) Consider the Laplace equation on the disk $0 \le r < 2$. Which of the listed functions would you try for a *bounded* solution $v = c_1v_1 + c_2v_2$? You are asked to list possible values for v_1 and v_2, not to find c_1 and c_2.

c) Find coefficients c_1 and c_2 such that $v = c_1v_1 + c_2v_2$ is a solution of the Laplace equation on the disk $0 \le r < 2$, subject to the boundary condition $v(2,\theta) = \cos(3\theta)$, $-\pi \le \theta < \pi$.

Problem 7: (18 points) Find the solution $u(x,y)$ of Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in the rectangle $0 < x < \pi$, $0 < y < 1$, that satisfies the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial x}(\pi,y) = 0, \qquad 0 < y < 1,$$
$$u(x,0) = 0, \qquad u(x,1) = 1 + \cos(5x), \qquad 0 < x < \pi.$$

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$