Homework 12 (due 12/03)

MAT 324: Real Analysis

Problem 1.

(i) Let a_n, $n \in \mathbb{N}$, be a non-negative sequence. Define a function $\mu\colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$ by $\mu(\emptyset) = 0$ and

$$\mu(E) = \sum_{n \in E} a_n$$

for $E \neq \emptyset$. Show that E is a σ-finite measure on $\mathbb{N}$.

(ii) Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be a measure space. Show that there exists a sequence a_n, $n \in \mathbb{N}$, of non-negative numbers such that $\mu(E) = \sum_{n \in E} a_n$ for all $E \neq \emptyset$.

Problem 2. Let μ, ν be two measures on the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. By Problem 1, there exist sequences $a_n, b_n \geq 0$, $n \in \mathbb{N}$, such that $\mu(E) = \sum_{n \in E} a_n$ and $\nu(E) = \sum_{n \in E} b_n$ for all $E \neq \emptyset$.

(i) What are necessary and sufficient conditions on the sequences a_n, b_n so that $\nu \ll \mu$?

(ii) If $\nu \ll \mu$, compute the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Problem 3. Suppose that μ, ν are two measures on a measure space $(\Omega, \mathcal{F})$.

(i) Suppose that μ and ν are finite measures. Show that $\nu \ll \mu$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $F \in \mathcal{F}$ with $\mu(F) < \delta$ we have $\nu(F) < \varepsilon$.

(ii) Show that the above statement does not hold in general without the assumption that the measures are finite.

Hint: Suppose that the (ε, δ) condition fails. Then there exists $\varepsilon > 0$ and sets F_n such that for all $n \in \mathbb{N}$ we have $\mu(F_n) < 1/2^n$ and $\nu(F_n) \geq \varepsilon$. Define

$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} F_k$ and compute $\mu(A)$ and $\nu(A)$. Where is the assumption of the finiteness of the measures used?

For (ii) let μ be the Lebesgue measure on $\mathbb{R}$ and construct a measure $\nu = \mu_h$ for some appropriate function $h \geq 0$.

Problem 4. Suppose that λ, ν, μ are σ-finite measures on a measurable space $(\Omega, \mathcal{F})$ with $\lambda \ll \mu$ and $\nu \ll \mu$.

(i) Prove that $\lambda + \nu \ll \mu$.
Hence, by the Radon-Nikodym theorem the derivatives $\frac{d\lambda}{d\mu}$, $\frac{d\nu}{d\mu}$, and $\frac{d(\lambda+\nu)}{d\mu}$ exist.

(ii) Show that

$$\frac{d(\lambda + \nu)}{d\mu} = \frac{d\lambda}{d\mu} + \frac{d\nu}{d\mu}$$

almost everywhere with respect to the measure μ; that is, the set where the above equality fails has μ-measure zero. Equivalently, we say that the above equality holds μ-a.e.

(iii) If $\lambda \ll \nu$, then show the "chain rule":

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}$$

almost everywhere with respect to μ.

(iv) If $\mu \ll \nu$, then show that

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu}\right)^{-1}$$

almost everywhere with respect to μ.