Homework 11 (due 11/21)

MAT 324: Real Analysis

Problem 1. Let $G \subset \mathbb{R}$ be a null set, i.e., $m_1(G) = 0$, and let $E \subset \mathbb{R}$ be any set (not necessarily measurable). Show that $G \times E$ is a null set, i.e., $m_2(G \times E) = 0$.
Hint: it suffices to show that $G \times [-n, n]$ is null for each $n \in \mathbb{N}$; explain why.

Problem 2.

(i) Suppose that $E \subset \mathbb{R}$ is measurable. Show that $E \times \mathbb{R}$ is also measurable, i.e., it lies in $\mathcal{M}_2$. Note that the same proof will show that $\mathbb{R} \times E$ is measurable.

(ii) Suppose that $E, F \subset \mathbb{R}$ are measurable. Show that $E \times F$ is measurable.

(iii) Suppose that $E, F \subset \mathbb{R}$ are measurable. Show that $m_2(E \times F) = m_1(E)m_1(F)$.

Hint: for (i) write E as $G \setminus N$, where G is a G_δ set and N is a null set. Then prove the statement for G and N, using Problem 1. For (ii) use part (i). For (iii) use Tonelli's theorem.

Problem 3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is measurable. Show that the function $\widetilde{f}(x, y) = f(x)$ is measurable as a function from $\mathbb{R}^2$ to $\mathbb{R}$.

Problem 4. Suppose that $f : \mathbb{R} \to [0, \infty)$ is a measurable function. Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq f(x)\}.$$

This is the region under the graph of f.

(i) Show that the set A is measurable.
Hint: express the set A in terms of functions defined on $\mathbb{R}^2$ and use Problem 3.

(ii) Show that

$$\int f(x)dx = m_2(A).$$

Hint: Tonneli's theorem.