# Homework 10 (due 11/14)

MAT 324: Real Analysis

**Problem 1.** Let $\mathcal{F}_1, \mathcal{F}_2$ be two $\sigma$-algebras in $\mathbb{R}$. Consider the $\sigma$-algebra $\mathcal{F}$ in $\mathbb{R}^2$ that is generated by sets of the form $A \times B$, where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Recall that, by definition, any other $\sigma$-algebra $\mathcal{C}$ that contains all sets of the form $A \times B$, where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, necessarily contains $\mathcal{F}$. The $\sigma$-algebra $\mathcal{F}$ is denoted by $\mathcal{F}_1 \times \mathcal{F}_2$.

Prove that if a set $E \subset \mathbb{R}^2$ is contained in the $\sigma$-algebra $\mathcal{F}$ then:

(i) for every $x \in \mathbb{R}$ the slice $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$ is contained in $\mathcal{F}_2$ and

(ii) for every $y \in \mathbb{R}$ the slice $E^y = \{x \in \mathbb{R} : (x, y) \in E\}$ is contained in $\mathcal{F}_1$.

*Hint:* Consider the collection $\mathcal{C}$ of sets satisfying (i) and (ii) and show that $\mathcal{C}$ is a $\sigma$-algebra containing all sets of the form $A \times B$, where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$.

**Problem 2.** Let $\mathcal{N} \subset [0, 1]$ be a non-measurable set and $C \subset [0, 1]$ be the Cantor set. Decide whether the following are true or false and explain your answer:

(i) $\mathcal{N} \times C$ is a Borel set.

(ii) $\mathcal{N} \times C$ lies in the $\sigma$-algebra $\mathcal{M}_1 \times \mathcal{M}_1$, where $\mathcal{M}_1$ is the $\sigma$-algebra of measurable subsets of $\mathbb{R}$.

(iii) $\mathcal{N} \times C$ lies in the $\sigma$-algebra $\mathcal{M}_2$ of measurable subsets of $\mathbb{R}^2$.

(iv) $\mathcal{M}_2 = \mathcal{M}_1 \times \mathcal{M}_1$.

*Hint:* use Problem 1.

**Problem 3.** Consider the function

$$g(x,y) = \begin{cases} \frac{1}{x^2}, & 0 < y < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\int g dm_2$.
*Hint:* use the criterion of Homework 8, Problem 5(i), which also holds for the 2-dimensional Lebesgue measure; you do not need to prove that.

**Problem 4.** Consider the function

$$g(x,y) = \begin{cases} \frac{1}{x^2} & 0 < y < x < 1 \\ -\frac{1}{y^2} & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that $\int_0^1 (\int_0^1 g(x,y)dx)dy = -1$ and $\int_0^1 (\int_0^1 g(x,y)dy)dx = 1$.

(ii) Explain why Fubini's theorem does not apply for the function $g$.

*Hint:* use Problem 3.