Homework 9 (due 11/07)

MAT 324: Real Analysis

Problem 1. Show Hölder's inequality when $p = 1$ and $q = \infty$, so that $\frac{1}{p} + \frac{1}{q} = 1$. More specifically, show that if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then $f \cdot g \in L^1(\mathbb{R})$ and

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

Problem 2. Let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Consider functions $f, f_n \in L^p(\mathbb{R})$ and $g, g_n \in L^q(\mathbb{R})$, $n \in \mathbb{N}$, such that $f_n \to f$ in L^p and $g_n \to g$ in L^q. Show that $f_n g_n \to fg$ in L^1.

Problem 3. Let $0 < p < \infty$ and define

$$f(x) = \begin{cases} 0, & x \leq 0 \\ x^{-1/p}(1 + |\log x|)^{-2/p}, & x > 0. \end{cases}$$

Show that $f \in L^p \setminus L^q$ for all $q \neq p$.

Problem 4.

(i) Suppose that $f \in L^p(E)$ for all $1 \leq p < \infty$. Is is true that $f \in L^\infty(E)$?

(ii) Suppose that $f \in L^\infty(E)$. Is is true that $f \in L^p(E)$ for any $p \geq 1$?

(iii) Suppose that $f \in L^\infty(E) \cap L^q(E)$ for some $q \geq 1$. Then show that $f \in L^p(E)$ for all $p > q$.

(iv) Suppose that $f \in L^\infty(E)$ and $m(E) < \infty$. Then show that $f \in L^p(E)$ for all $1 \leq p < \infty$.

Problem 5.

(i) Suppose that f is continuous in $\mathbb{R}$ and lies in L^p for some $p \geq 1$. What can you say about $\lim_{x\to\infty} f(x)$?

(ii) Suppose that f is uniformly continuous in $\mathbb{R}$ and lies in L^p for some $p \geq 1$. What can you say about $\lim_{x\to\infty} f(x)$?

Problem 6 (Optional)**.** Show that if $f \in L^\infty(\mathbb{R}) \cap L^q(\mathbb{R})$ for some $q \geq 1$, then $f \in L^p(\mathbb{R})$ for all $p > q$. Moreover, show that

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$