Homework 8 (due 10/31)

MAT 324: Real Analysis

Problem 1. Let $(H, (\cdot,\cdot))$ be an inner product space. Show the polarization identity:

$$4(f,g) = \|f+g\|^2 - \|f-g\|^2 + i(\|f+ig\|^2 - \|f-ig\|^2)$$

for all $f, g \in H$.

Problem 2. Consider the space $L^1([0,1])$. Show that there is no inner product on the space $L^1([0,1])$ that gives rise to the L^1 norm.
Hint: show that the parallelogram law is not satisfied for some pair of functions.

Problem 3. Suppose that $m(E) < \infty$. Show that $L^2(E) \subset L^1(E)$ and show that the reverse inclusion does not hold.

Problem 4. Let $f_n \in L^1([0,1]) \cap L^2([0,1])$, $n \in \mathbb{N}$. Prove or disprove the following:

(i) If $\|f_n\|_{L^1} \to 0$ then $\|f_n\|_{L^2} \to 0$.

(ii) If $\|f_n\|_{L^2} \to 0$ then $\|f_n\|_{L^1} \to 0$.

Problem 5.

(i) Let $f\colon E \to \mathbb{R}$ be a measurable function. Show that $f \in L^p(E)$, where $1 \le p < \infty$, if and only if

$$\sum_{k=-\infty}^{\infty} 2^{kp} m(\{2^k \le |f| < 2^{k+1}\}) < \infty.$$

(ii) For $\alpha \in \mathbb{R}$ consider the function $f(x) = |x|^{-\alpha} \cdot \mathbf{1}_{(0,1)}$. For each $p \in [1,\infty)$ specify the set of $\alpha \in \mathbb{R}$ for which $f \in L^p(\mathbb{R})$.

Problem 6. Let $p \in [1,\infty)$. Suppose that $f_n \in L^p(E)$, $n \in \mathbb{N}$, is a sequence of functions converging to a function $f \in L^p(E)$ in the L^p norm. Show that there exists a subsequence of f_n that converges to f almost everywhere.
Hint: follow the proof of the case $p = 1$ that we did in class.