Homework 7 (due 10/17)

MAT 324: Real Analysis

Problem 1. Suppose that $f_n \in L^1(\mathbb{R})$, $n \in \mathbb{N}$, and $|f_n| \leq g$ for all $n \in \mathbb{N}$, where $g \in L^1(\mathbb{R})$. Show that

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n.$$

Conclude that if f_n converges almost everywhere to a function f, then $\int f = \lim_{n\to\infty} \int f_n$.
Remark: the first inequality holds always for *non-negative* functions by Fatou's lemma. However, the last inequality does not necessarily hold even for non-negative functions. Why?

Problem 2. Let $E \subset [0,1]$ be a Cantor-like set of positive measure (see Homework 2, Problem 8) and set $f = \mathbf{1}_E$. (The only properties of E we are going to need are that it has positive measure, it is compact and it contains no intervals).

(i) Show that f is not Riemann integrable on $[0,1]$.

(ii) Let $d(x,E) = \inf\{|x-y| : y \in E\}$. Show that the function $x \mapsto d(x,E)$ is continuous on $[0,1]$.

(iii) Let $f_n(x) = \max(0, 1-nd(x,E))$. Show that f_n, $n \in \mathbb{N}$, is a decreasing sequence of continuous functions converging pointwise to f.

(iv) Show that f_n, $n \in \mathbb{N}$, is a Cauchy sequence in the L^1 metric; that is for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$\int |f_n - f_m| < \varepsilon.$$

(v) Conclude that the metric space of equivalence classes of Riemann integrable functions on $[0,1]$ with the L^1 metric is not complete.

Problem 3. Consider the sequence of functions $f_n(x) = \frac{1}{\sqrt{x}}\mathbf{1}_{(0,1/n]}(x)$, $n \in \mathbb{N}$.

(i) Does f_n lie in $L^1(\mathbb{R})$? Check this without using the fundamental theorem of calculus, but using instead dyadic intervals, i.e., intervals of the form $[2^{-k}, 2^{-k+1}]$, $k \in \mathbb{N}$.

(ii) Is f_n, $n \in \mathbb{N}$, a Cauchy sequence in $L^1(\mathbb{R})$?

(iii) For which $p \in [1, \infty)$ do we have

$$\int |f_n|^p < \infty?$$

Problem 4. Consider the sequence of functions $f_n = n\mathbf{1}_{[n+1/n^2, n+2/n^2]}$, $n \geq 2$.

(i) Show that the infinite series $\sum_{n=2}^{\infty} f_n$ converges pointwise to a function f.

(ii) Does the series $\sum_{n=2}^{\infty} f_n$ converge in L^1 to f?

(iii) Show that $\sum_{n=2}^{\infty} f_n^{1/2}$ converges in L^1 to $f^{1/2}$.

Problem 5. Let $f\colon \mathbb{R} \to [0, \infty]$ be a measurable function. Consider the *distribution function* of f, defined by

$$\lambda_f(t) = m(\{x \in \mathbb{R} : f(x) > t\}).$$

Prove that:

(i) λ_f is a decreasing function of $t \in \mathbb{R}$ and it is right-continuous.

(ii) If $0 \leq f \leq g$, then $\lambda_f \leq \lambda_g$.

(iii) If f_n, $n \in \mathbb{N}$, is a sequence of non-negative measurable functions increasing to f, then λ_{f_n} is an increasing sequence of functions, converging to λ_f pointwise.

(iv)
$$\int_{\mathbb{R}} f dm = \int_0^{\infty} \lambda_f(t) dt.$$

Hint: show part (iv) first for simple functions. Then use approximation and convergence theorems to show the statement for any non-negative measurable function.

Problem 6. Let $(X, \|\cdot\|)$ be a normed vector space. Show that X is complete if and only if the following holds: whenever $x_n \in X$, $n \in \mathbb{N}$, is a sequence with $\sum_{n=1}^{\infty} \|x_n\| < \infty$, we have that the series $\sum_{n=1}^{\infty} x_n$ converges to an element $x \in X$; that is $\lim_{N\to\infty} \|\sum_{n=1}^{N} x_n - x\| = 0$ for some $x \in X$.

Problem 7 (Optional)**.** Does there exist a function $f\colon \mathbb{R} \to [0, \infty)$ with $f(x) < \infty$ for all $x \in \mathbb{R}$, but

$$\int_a^b f = \infty$$

for all intervals $[a, b] \subset \mathbb{R}$?

Problem 8 (Optional)**.** Does there exist a function $f\colon \mathbb{R} \to [0, \infty)$ that is unbounded in every interval, but

$$\int_a^b f < \infty$$

for all intervals $[a, b] \subset \mathbb{R}$?

Problem 9 (Optional)**.**

(i) If $f \in L^1(\mathbb{R})$, is it true that $\lim_{x\to\infty} f(x) = 0$?

(ii) If $f \in L^1(\mathbb{R})$ is continuous and bounded, is it true that $\lim_{x\to\infty} f(x) = 0$?