Homework 6 (due 10/10)

MAT 324: Real Analysis

Problem 1. Suppose that $f, g \in L^1(\mathbb{R})$ are such that

$$\int_E f dm = \int_E g dm$$

for all $E \in \mathcal{M}$. Show that $f = g$ a.e.

Problem 2. Let $f\colon \mathbb{R} \to [0, \infty)$ be a continuous function that is not identically equal to 0. Show that

$$\int_{\mathbb{R}} f dm > 0.$$

Problem 3. Let $E \in \mathcal{M}$ and $f_n, f, g\colon E \to \mathbb{R}$ be measurable functions, $n \in \mathbb{N}$. Show that following "almost everywhere" versions of the monotone and dominated convergence theorems:

(i) If $f_n \geq 0$ for all $n \in \mathbb{N}$ and f_n increases to f pointwise a.e. in E, then

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm.$$

(ii) If $g \in L^1(E)$, $|f_n| \leq g$ a.e. in E, and f_n converges to f pointwise a.e. in E, then $f \in L^1(E)$ and

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm.$$

Hint for (ii): for each n, there exists a null set G_n such that $|f_n| \leq g$ in $E \backslash G_n$. Consider the set $E \setminus \bigcup_{n\in\mathbb{N}} G_n$ and apply the usual dominated convergence theorem to an appropriate sequence of functions.

Problem 4. Show the following stronger version of the dominated convergence theorem. Let $E \in \mathcal{M}$ and $f_n, f, g\colon E \to \mathbb{R}$ be measurable functions, $n \in \mathbb{N}$, such that $g \in L^1(E)$, $|f_n| \leq g$ a.e. in E, and $f_n \to f$ pointwise a.e. in E. Then $f \in L^1(E)$ and

$$\lim_{n\to\infty} \int_E |f_n - f| dm = 0.$$

Note that this implies that $\lim_{n\to\infty} \int_E f_n dm = \int_E f dm$. (How?)
Hint: set $F_n = |f_n - f|$ on E and then use the previous problem to this sequence.

Problem 5. Suppose that $f_n, f\colon \mathbb{R} \to [0, \infty]$ are measurable functions, $n \in \mathbb{N}$, such that f_n decreases to f. Show that in general

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n dm \neq \int_{\mathbb{R}} f dm.$$

(That is, give an example where this fails.) Give sufficient conditions so that we do have equality above (and also prove why your conditions suffice).
Hint: what assumption did we need in order to conclude that $m(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} m(E_n)$ for a decreasing sequence of sets E_n?

Problem 6. Suppose that $E_n \in \mathcal{M}$, $n \in \mathbb{N}$, are disjoint sets and $E = \bigcup_{n\in\mathbb{N}} E_n$. If $f \in L^1(E)$, show that

$$\sum_{n=1}^{\infty} \int_{E_n} f dm = \int_E f dm.$$

Problem 7. Let $f \in L^1(\mathbb{R})$ and $E_n = \{|f| \leq n\}$. Show that

$$\int_{\mathbb{R}} f dm = \lim_{n\to\infty} \int_{[-n,n]} f dm = \lim_{n\to\infty} \int_{E_n} f dm = \lim_{n\to\infty} \int_{E_n \cap [-n,n]} f dm.$$

Remark: Note that $f \cdot 1_{E_n \cap [-n,n]}$ is a bounded function, with a bounded domain. Many times it is much easier to prove theorems for such functions, rather than prove them directly for all integrable functions. The equalities above allow us to "go back and forth" between integrable functions and functions with bounded domain and range.

Problem 8. Compute the limit

$$\lim_{n\to\infty} \int_0^n (1 - x/n)^n dx.$$

Justify carefully your calculation.