Homework 5 (due 10/03)

MAT 324: Real Analysis

Problem 1. Show that for simple functions the two definitions of the Lebesgue integral that we gave agree. Namely, if $f = \sum_{i=1}^n a_i 1_{A_i}$ is a simple function expressed in canonical representation, then $\int_{\mathbb{R}} f dm$ can be defined in the following two ways:

$$\int_{\mathbb{R}} f dm = \sum_{i=1}^{n} a_i m(A_i)$$

and

$$\int_{\mathbb{R}} f dm = \sup \left\{ \int_{\mathbb{R}} \phi dm : 0 \leq \phi \leq f \text{ and } \phi \text{ is simple} \right\}.$$

Show that the right hand sides agree.

Problem 2. Show that a simple function $\phi \colon \mathbb{R} \to [0, \infty)$ is integrable if and only if $m(\{\phi \neq 0\}) < \infty$.

Problem 3. Define a function $f \colon [0,1] \to \mathbb{R}$ as follows. Set $f(x) = 0$ for all x in the Cantor set, and if x lies in an interval of length 3^{-k} that has been removed from $[0,1]$ in the construction of the Cantor set, we set $f(x) = k$. Show that f is measurable and evaluate $\int_{[0,1]} f dm$.

Problem 4. Let $f_n, f \colon E \to [0, \infty]$ be measurable functions and assume that f is integrable and $m(E) < \infty$. Suppose that the sequence f_n converges uniformly to f. Show that f_n is integrable for all sufficiently large $n \in \mathbb{N}$ and that

$$\lim_{n\to\infty} \int_E f_n dm = \int_E \lim_{n\to\infty} f_n dm = \int_E f dm.$$

Does the conclusion hold if we drop the assumption that $m(E) < \infty$?
Remark: The statement holds not only for non-negative but also for $\overline{\mathbb{R}}$-valued functions.

Problem 5. Let $f\colon \mathbb{R} \to [0, \infty]$ be a measurable function and let $\alpha > 0$. Show that

$$m(\{x \in \mathbb{R} : f(x) > \alpha\}) \leq \frac{1}{\alpha} \int_{\mathbb{R}} f dm.$$

This is known as Chebychev's inequality.

Moreover, show that the factor $1/\alpha$ in the above inequality is optimal, in the following sense: if $C(\alpha) > 0$ is a positive function of $\alpha > 0$ such that

$$m(\{x \in \mathbb{R} : f(x) > \alpha\}) \leq C(\alpha) \int_{\mathbb{R}} f dm$$

for all non-negative measurable functions f and for all $\alpha > 0$, then $1/\alpha \leq C(\alpha)$.

Problem 6. Suppose that $f\colon \mathbb{R} \to [0, \infty]$ is integrable. Show then that f is finite a.e. Does the converse hold? Namely, if f is finite a.e., is it true that it is integrable?
Hint: Find bounds for the measure of the set $\{f = \infty\}$ using Chebychev's inequality.