# Homework 4 (due 09/26)

MAT 324: Real Analysis

**Problem 1.** Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and let $f\colon E \to \mathbb{R}$ be measurabule, where $E \in \mathcal{M}$. Show that the function $\phi \circ f$ is measurable.
Remark: In particular, $\sin f(x)$, $|f|, \dots$ are measurable.

**Problem 2.** Suppose that $f\colon E \to \overline{\mathbb{R}}$ is measurable, where $E \in \mathcal{M}$ is a set of finite measure. If the function $f$ is finite almost everywhere, then show that for each $\varepsilon > 0$ there exists $M > 0$ such that $m(\{|f| > M\}) < \varepsilon$.
Does the statement hold if $E$ is infinite measure?

**Problem 3.** Suppose that $f, g\colon E \to \mathbb{R}$ are measurable functions, where $E \in \mathcal{M}$.

(i) If $f = g$ a.e. on $E$, and $g$ is continuous, is it true that $f$ is continuous a.e. on $E$?

(ii) If $f = g$ a.e. on $E$, $g$ is continuous on $E$, and $f$ is a.e. continuous on $E$, is it true that $f$ is continuous?

(iii) If $f = g$ a.e. on $\mathbb{R}$, and both $f, g$ are continuous, is it true that $f = g$ on $\mathbb{R}$?

**Problem 4.** Suppose that the function $f\colon (a, b) \to \mathbb{R}$ is differentiable at each $x \in (a, b)$. Show that the derivative $f'\colon (a, b) \to \mathbb{R}$ is measurable.

**Problem 5.** Show Egorov's theorem with the weaker assumption that $f_n \to f$ a.e. on $E$.

**Problem 6.** Show that in Egorov's theorem the set $E \setminus A$, where $f_n$ converges uniformly, can be taken to be compact.

**Problem 7** (Optional)**.** Let $f\colon [a, b] \to \mathbb{R}$ be a bounded, Borel-measurable function. Show that the set $\{x \in [a, b] : f \text{ is continuous at } x\}$ is a Borel set.
Hint: Consider the functions $\overline{f}(x) := \limsup_{y\to x} f(y)$ and $\underline{f}(x) = \liminf_{y\to x} f(y)$.

Show that both of these functions are Borel-measurable and then observe that the set where $f$ is continuous is equal to the set where these two functions agree with each other and with $f$.

**Problem 8** (Optional)**.**

(i) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the set $\{x \in \mathbb{R} : f'(x) \text{ exists}\}$ is measurable.

(ii) Show the same conclusion, but this time assume that $f$ is merely measurable.

Hint: Study the function

$$g(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{n \to \infty} \sup_{0<h<1/n} \frac{f(x+h) - f(x)}{h}$$

and show that it is measurable using the definition of measurability.