Homework 3 (due 09/19)

MAT 324: Real Analysis

Problem 1. Show that the family of intervals of the form $[a, b)$, $a < b$, $a, b \in \mathbb{R}$, generates the Borel σ-algebra.

Problem 2. Show that a set $E \subset \mathbb{R}$ is measurable if and only if for each $\varepsilon > 0$ there exists an open set $\mathcal{O} \supset E$ such that $m^*(\mathcal{O} \setminus E) < \varepsilon$. (We have proved one direction in the lecture.)

Problem 3. Let $f\colon E \to \mathbb{R}$ be a function, where $E \in \mathcal{M}$. Suppose that for any two rational numbers p, q with $p < q$ the set $\{x \in E : p < f(x) < q\}$ is measurable. Show that f is measurable.

Problem 4. Let $f\colon E \to \mathbb{R}$ be a function, where $E \in \mathcal{M}$. For $a \in \mathbb{R}$ consider the level set $E_a \coloneqq \{x \in E : f(x) = a\}$.

(i) Show directly that if f is measurable then for each $a \in \mathbb{R}$ the set E_a is measurable.

(ii) Suppose that E_q is measurable for each $q \in \mathbb{Q}$. Does it follow that the function f is measurable?

Problem 5. Suppose that a function $f\colon \mathbb{R} \to \mathbb{R}$ is increasing; that is if $x < y$, $x, y \in \mathbb{R}$, then $f(x) \leq f(y)$. Show that f is measurable.

Problem 6.

(i) Let $E \in \mathcal{M}$ and $f\colon E \to \mathbb{R}$ be a function. Suppose that f is continuous on $E \setminus N$, where $m(N) = 0$; that is, f is continuous *almost everywhere on* E. Show that f is measurable.

(ii) Using part (i) and the fact that an increasing function is continuous, except at a countable set, give an alternative proof of Problem 5.

Problem 7.

(i) Let $E \in \mathcal{M}$ and $f, g\colon E \to \mathbb{R}$ be measurable functions. Show that the function f/g, defined on the set $F = \{x \in E : g(x) \neq 0\}$ is measurable. (Part of the proof is to explain why the set F is measurable.)

(ii) Let $h\colon \mathbb{R} \to \mathbb{R}$ be a non-measurable function and $f\colon \mathbb{R} \to \mathbb{R}$ be a measurable function such that the set $\{x \in \mathbb{R} : f(x) = 0\}$ is a null set; that is, $f \neq 0$ *almost everywhere*. Decide whether the function $f \cdot h$ is measurable or not.

Problem 8 (Optional)**.** Let $E \in \mathcal{M}$ with $m(E) > 0$. Consider the set $E - E = \{x - y : x, y \in E\}$. Show that $E - E$ contains an interval of the form $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.
Hint: Consider the function $f(x) = m((E + x) \cap E)$, $x \in \mathbb{R}$. Show that f is continuous; show first that this holds if E is an interval or a finite union of disjoint intervals. Then show that $f(0) > 0$ and conclude that there exists an interval $(-\varepsilon, \varepsilon)$ such that $f(x) > 0$ for all $x \in (-\varepsilon, \varepsilon)$.

Problem 9 (Optional)**.** Let $E \in \mathcal{M}$ with $m(E) > 0$. Show that there exists a non-measurable set $N \subset E$.
Hint: Use the construction of a non-measurable set and Problem 8.

Problem 10 (Optional)**.** The goal of this problem is to show that $\mathcal{B} \subsetneq \mathcal{M}$. Let $f\colon [0,1] \to [0,1]$ be a continuous, increasing and surjective function with the property that $f(\mathcal{C}) = [0,1]$, where $\mathcal{C}$ is the Cantor set; for example, such a function is the Cantor staircase function. Consider the function $g(x) = f(x) + x$, $x \in [0,1]$.

(i) Show that $g\colon [0,1] \to [0,2]$ is continuous, strictly increasing, one-to-one, onto, and that g^{-1} is continuous.

(ii) Show that $g(\mathcal{C})$ is measurable and $m(g(\mathcal{C})) = 1$.
Hint: Study the set $g([0,1] \setminus \mathcal{C})$.

(iii) Let $N \subset g(\mathcal{C})$ be a non-measurable set, guaranteed to exist by Problem 9. Show that $g^{-1}(N) \in \mathcal{M}$.

(iv) Show that $A \coloneqq g^{-1}(N) \notin \mathcal{B}$. Do this by showing that a continuous, injective function $h\colon [a,b] \to \mathbb{R}$ has the property that $h(A) \in \mathcal{B}$ whenever $A \subset [a,b]$, $A \in \mathcal{B}$.