Homework 2 (due 09/12)

MAT 324: Real Analysis

Problem 1.

(i) Let $E = \bigcup_{n=1}^{\infty} E_n$. Show that $m^*(E) = 0$ if and only if $m^*(E_n) = 0$ for all $n \in \mathbb{N}$.

(ii) Show that the outer measure is *translation invariant*: for each $A \subset \mathbb{R}$ and $t \in \mathbb{R}$ we have

$$m^*(A) = m^*(A + t),$$

where $A + t = \{x + t : x \in A\}$.

Solution. (i) If $m^*(E_n) = 0$ for all $n \in \mathbb{N}$, then by the subadditivity we have

$$m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n) = 0.$$

If $m^*(E) = 0$, then by the monotonicity, since $E_n \subset E$, we have $m^*(E_n) \leq m^*(E) = 0$ for each $n \in \mathbb{N}$

(ii) If $\{I_n\}_{n\in\mathbb{N}}$ are intervals such that $A \subset \bigcup_{n=1}^{\infty} I_n$, then the intervals $\{I_n + t\}_{n\in\mathbb{N}}$ cover $A + t$. Hence,

$$m^*(A + t) \leq \sum_{n=1}^{\infty} \ell(I_n + t) = \sum_{n=1}^{\infty} \ell(I_n).$$

This holds for all intervals $\{I_n\}_{n\in\mathbb{N}}$ as above so by the definition of outer measure we have $m^*(A+t) \leq m^*(A)$. The same argument shows the reverse inequality. $\square$

Problem 2.

(i) Show that a countable intersection of measurable sets is measurable.

(ii) Suppose that $A, B \subset \mathbb{R}$ are such that $m^*(A\Delta B) = 0$. Here, $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Show that $A \in \mathcal{M}$ if and only if $B \in \mathcal{M}$.

Solution. (i) Let $E_n \in \mathcal{M}$, $n \in \mathbb{N}$. Then

$$\bigcap_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} E_n^c \right)^c$$

and the latter is measurable (why?).

(ii) Let $G \subset \mathbb{R}$ be an arbitrary set and suppose that $A \in \mathcal{M}$, so

$$m^*(G) = m^*(G \cap A) + m^*(G \cap A^c).$$

We wish to show that

$$m^*(G) = m^*(G \cap B) + m^*(G \cap B^c).$$

Using the assumption, one can show (how?) that $m^*(G \cap B) = m^*(G \cap A)$ and $m^*(G \cap B^c) = m^*(G \cap A^c)$ so the conclusion follows. □

Problem 3. Show that if $A, B \in \mathcal{M}$ with $A \subset B$ and $m(A) < \infty$, then

$$m(B \setminus A) = m(B) - m(A).$$

Does the statement hold if $m(A) = \infty$?

Solution. Note that $B = (B \setminus A) \cup A$, because $A \subset B$. By the additivity of the measure, we have

$$m(B) = m(B \setminus A) + m(A).$$

Since $m(A) < \infty$, we have $m(B \setminus A) = m(B) - m(A)$; note that this even holds if $m(B) = \infty$. We only have to avoid having expressions of the form $\infty - \infty$. Indeed, if $m(A) = \infty$ then the statement fails. For example, let $A = [0, \infty)$ and $B = \mathbb{R}$. Then $m(B) - m(A)$ does not make sense. □

Problem 4. Suppose that $A, B \in \mathcal{M}$. Show that

$$m(A \cup B) + m(A \cap B) = m(A) + m(B).$$

Solution. Note that $A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B)$. All sets involved are measurable and disjoint, so by the additivity of the measure we have

$$m(A \cup B) = m(A \setminus (A \cap B)) + m(B \setminus (A \cap B)) + m(A \cap B).$$

If $m(A \cap B) = \infty$ then $m(A) \geq m(A \cap B) = \infty$ so the required statement holds and we have nothing to prove. Suppose that $m(A \cap B) < \infty$. Using the previous problem we have

$$\begin{aligned} m(A \cup B) &= m(A) - m(A \cap B) + m(B) - m(A \cap B) + m(A \cap B) \\ &= m(A) + m(B) - (A \cap B) \end{aligned}$$

as desired. □

Problem 5. Let $E_n \in \mathcal{M}$, $n \in \mathbb{N}$. Is it true that

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} m(E_n)?$$

What if we further assume that $E_n \subset E_{n+1}$ for each $n \in \mathbb{N}$? What if we even further assume that the limit in the right hand side exists and is a finite number?

Solution. In general the statement is not true. For example, let $E_n = [n, n+1]$. Then the intersection is empty, but $m(E_n) = 1$.

Even if we assume that $E_n \subset E_{n+1}$ the statement fails. For example, let $E_n = [n, \infty)$. Then $\bigcap_{n=1}^{\infty} E_n = \emptyset$ so its measure is 0. On the other hand, $m(E_n) = \infty$, so $\lim_{n \to \infty} m(E_n) = \infty \neq 0$.

Finally, if we assume that $\lim_{n \to \infty} m(E_n) < \infty$, then the statement is true. See the proof of Theorem 2.19(ii) in the book. □

Problem 6. Let $E \in \mathcal{M}$. Show that

$$m(E) = \sup\{m(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

Solution. We have proved that if $E \in \mathcal{M}$, then for each $\varepsilon > 0$ there exists an open set $\mathcal{O} \supset E$ such that $m(\mathcal{O} \setminus E) < \varepsilon$.

Since $E \in \mathcal{M}$, we have $E^c \in \mathcal{M}$. Hence, for each ε there exists an open set $\mathcal{O} \supset E^c$ such that $m(\mathcal{O} \setminus E^c) < \varepsilon$. Note that $\mathcal{O} \setminus E^c = \mathcal{O} \cap E = E \setminus \mathcal{O}^c$. Therefore, we have $m(E \setminus K) < \varepsilon$. The set $K := \mathcal{O}^c$ is closed and is contained in E (why?). It follows that for each $\varepsilon > 0$, there exists a closed set $K \subset E$ such that $m(K) \leq m(E) = m(E \setminus K) + m(K) \leq \varepsilon + m(K)$. This shows (why?) that

$$m(E) = \sup\{m(K) : K \subset E \text{ and } K \text{ is closed}\}.$$

Finally, we want to replace "closed" by "compact". Let $\varepsilon > 0$ and consider a closed set $K \subset E$ with $m(K) \leq m(E) \leq m(K) + \varepsilon/2$. For each

$n \in \mathbb{N}$, define $K_n = K \cap [-n, n]$ and note that K_n is compact (why?). Also, observe that $K_n \subset K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = K$. Therefore by Theorem 2.19(i) we have

$$m(K) = \lim_{n \to \infty} m(K_n).$$

In case $m(E) < \infty$, then $m(K) < \infty$ (why?), so this implies that there exists a sufficiently large n such that $m(K) \leq m(K_n) + \varepsilon/2$. Therefore, $K_n \subset K \subset E$ and

$$m(K_n) \leq m(K) \leq m(E) \leq m(K) + \varepsilon/2 \leq m(K_n) + \varepsilon/2 + \varepsilon/2 = m(K_n) + \varepsilon.$$

Thus, we have "approximated" the measure of E by the measure of a compact set K_n, i.e.,

$$m(E) = \sup\{m(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

If $m(E) = \infty$, then we must also have $m(K) = \infty$ (why?). If it follows that $\lim_{n \to \infty} m(K_n) = \infty$, so $m(K_n)$ becomes arbitrarily large as n increases, and in particular it approaches the measure of E. It also follows in this case that

$$m(E) = \infty = \sup\{m(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

(Recall that $\sup Z = \infty$ if and only if there exists a sequence $z_n \in Z$ with $\lim_{n \to \infty} z_n = \infty$.) □

Problem 7. Construct a *Cantor-like* set $\mathcal{C}(\alpha) \subset [0, 1]$ as follows. Fix a number $\alpha \in (0, 1)$ and let $\mathcal{C}_0 = [0, 1]$. In order to obtain the set $\mathcal{C}_1$, remove from $\mathcal{C}_0$ the "middle" open interval of length α; for example, if $\alpha = 1/3$ as in the standard Cantor set, then we remove $(1/3, 2/3)$. We write $\mathcal{C}_1 = I_1^1 \cup I_2^1$. From each of the intervals I_i^1, $i = 1, 2$, we remove a "middle" open interval of length $\alpha \cdot \ell(I_i^1)$ and obtain in this way the set $\mathcal{C}_2$. In the n-th step we have a set $\mathcal{C}_n$ that is the union of 2^n disjoint intervals I_i^n, $i = 1, \ldots, 2^n$, and in order to obtain $\mathcal{C}_{n+1}$ we remove from each of them a "middle" open interval of length $\alpha \cdot \ell(I_i^n)$. Let

$$\mathcal{C}(\alpha) = \bigcap_{n=0}^{\infty} \mathcal{C}_n.$$

Compute $m(C(\alpha))$.
Hint: Using induction, find a formula for $m([0, 1] \setminus \mathcal{C}_n)$ depending on α and n. Then pass to the limit.

Solution. At the first step, we see that $m(\mathcal{C}_1) = 1-\alpha$ and $m([0,1]\setminus\mathcal{C}_1) = \alpha$. Intuitively, at each step we are keeping a proportion $(1-\alpha)$ of the measure and we are throwing away a proportion α of the measure. Inductively, one can show (how?) that

$$m(\mathcal{C}_{n+1}) = (1-\alpha)m(\mathcal{C}_n)$$

for each $n \in \mathbb{N}$. Therefore, $m(\mathcal{C}_n) = (1-\alpha)^n$. Since $\mathcal{C}(\alpha) \subset \mathcal{C}_n$ for all $n \in \mathbb{N}$, we have

$$m(\mathcal{C}(\alpha)) \leq (1-\alpha)^n$$

for all $n \in \mathbb{N}$, which implies that $m(\mathcal{C}(\alpha)) = 0$.
Remark: Surprisingly, even if we throw away 1/1000-th of the set at every step, we still get a set of measure 0. □

Problem 8 (Optional)**.** Modify the construction of $\mathcal{C}(\alpha)$ of Problem 7 as follows. Instead of removing a fixed proportion α at each step, remove variable proportions. That is, C_1 is obtained from C_0 by removing a "middle" interval of length α_1. Then C_2 is obtained from C_1 by removing from each of the two intervals of C_1 a middle interval of proportion α_2, and so on. In this way, we obtain another *Cantor-like* set $\mathcal{C}(\{\alpha_n\}_{n\in\mathbb{N}})$ that depends on the sequence of proportions that we choose. Show that the proportions can be chosen so that this Cantor-like set has positive measure.

Problem 9 (Optional)**.** Let $\mathcal{O} \subset \mathbb{R}$ be an open set. Show that there exist *disjoint* open intervals I_i, $i \in \Lambda$, such that

$$\mathcal{O} = \bigcup_{i\in\Lambda} I_i.$$

Moreover, the index set Λ is finite or countable.

Problem 10 (Optional)**.** Let $E \in \mathcal{M}$ with $m(E) > 0$. Then for any $\alpha \in (0,1)$ there exists an interval I such that

$$m(E \cap I) > \alpha m(I).$$

Remark: If $\alpha = 0.999$, for example, this says that we can always "zoom in" the set E at a small interval I and see that E takes up a lot of space within that interval. An alternative formulation is $m(I\setminus E) < (1-\alpha)m(I) = 0.001m(I)$, so E covers almost all of I.