Homework 2 (due 09/12)

MAT 324: Real Analysis

Problem 1.

(i) Let $E = \bigcup_{n=1}^{\infty} E_n$. Show that $m^*(E) = 0$ if and only if $m^*(E_n) = 0$ for all $n \in \mathbb{N}$.

(ii) Show that the outer measure is *translation invariant*: for each $A \subset \mathbb{R}$ and $t \in \mathbb{R}$ we have

$$m^*(A) = m^*(A + t),$$

where $A + t = \{x + t : x \in A\}$.

Problem 2.

(i) Show that a countable intersection of measurable sets is measurable.

(ii) Suppose that $A, B \subset \mathbb{R}$ are such that $m^*(A \Delta B) = 0$. Here, $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Show that $A \in \mathcal{M}$ if and only if $B \in \mathcal{M}$.

Problem 3. Show that if $A, B \in \mathcal{M}$ with $A \subset B$ and $m(A) < \infty$, then

$$m(B \setminus A) = m(B) - m(A).$$

Does the statement hold if $m(A) = \infty$?

Problem 4. Suppose that $A, B \in \mathcal{M}$. Show that

$$m(A \cup B) + m(A \cap B) = m(A) + m(B).$$

Problem 5. Let $E_n \in \mathcal{M}$, $n \in \mathbb{N}$. Is it true that

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} m(E_n)?$$

What if we further assume that $E_{n+1} \subset E_n$ for each $n \in \mathbb{N}$? What if we even further assume that the limit in the right hand side exists and is a finite number?

Problem 6. Let $E \in \mathcal{M}$. Show that

$$m(E) = \sup\{m(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

Problem 7. Construct a *Cantor-like* set $\mathcal{C}(\alpha) \subset [0,1]$ as follows. Fix a number $\alpha \in (0,1)$ and let $\mathcal{C}_0 = [0,1]$. In order to obtain the set $\mathcal{C}_1$, remove from $\mathcal{C}_0$ the "middle" open interval of length α; for example, if $\alpha = 1/3$ as in the standard Cantor set, then we remove $(1/3, 2/3)$. We write $\mathcal{C}_1 = I_1^1 \cup I_2^1$. From each of the intervals I_i^1, $i = 1, 2$, we remove a "middle" open interval of length $\alpha \cdot \ell(I_i^1)$ and obtain in this way the set $\mathcal{C}_2$. In the n-th step we have a set $\mathcal{C}_n$ that is the union of 2^n disjoint intervals I_i^n, $i = 1, \dots, 2^n$, and in order to obtain $\mathcal{C}_{n+1}$ we remove from each of them a "middle" open interval of length $\alpha \cdot \ell(I_i^n)$. Let

$$\mathcal{C}(\alpha) = \bigcap_{n=0}^{\infty} \mathcal{C}_n.$$

Compute $m(C(\alpha))$.
Hint: Using induction, find a formula for $m([0,1] \setminus \mathcal{C}_n)$ depending on α and n. Then pass to the limit.

Problem 8 (Optional)**.** Modify the construction of $\mathcal{C}(\alpha)$ of Problem 7 as follows. Instead of removing a fixed proportion α at each step, remove variable proportions. That is, C_1 is obtained from C_0 by removing a "middle" interval of length α_1. Then C_2 is obtained from C_1 by removing from each of the two intervals of C_1 a middle interval of proportion α_2, and so on. In this way, we obtain another *Cantor-like* set $\mathcal{C}(\{\alpha_n\}_{n\in\mathbb{N}})$ that depends on the sequence of proportions that we choose. Show that the proportions can be chosen so that this Cantor-like set has positive measure.

Problem 9 (Optional)**.** Let $\mathcal{O} \subset \mathbb{R}$ be an open set. Show that there exist *disjoint* open intervals I_i, $i \in \Lambda$, such that

$$\mathcal{O} = \bigcup_{i\in\Lambda} I_i.$$

Moreover, the index set Λ is finite or countable.

Problem 10 (Optional)**.** Let $E \in \mathcal{M}$ with $m(E) > 0$. Then for any $\alpha \in (0,1)$ there exists an interval I such that

$$m(E \cap I) > \alpha m(I).$$

Remark: If $\alpha = 0.999$, for example, this says that we can always "zoom in" the set E at a small interval I and see that E takes up a lot of space within that interval. An alternative formulation is $m(I \setminus E) < (1-\alpha)m(I) = 0.001m(I)$, so E covers almost all of I.