Homework 1 (due 09/05)

MAT 324: Real Analysis

Problem 1. Let $A \subset \mathbb{R}$ be a countable set. Show that A is a null set by proving that for each $\varepsilon > 0$ there exist *open* intervals I_n, $n \in \mathbb{N}$, such that $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$.

Solution. Let $A = \{a_1, a_2, \ldots\}$ be an enumeration of A. Fix $\varepsilon > 0$ and define $I_n = (a_n - \varepsilon/2^{n+2}, a_n + \varepsilon/2^{n+2})$ for $n \in \mathbb{N}$. Then $A \subset \bigcup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} 2\frac{\varepsilon}{2^{n+2}} = \varepsilon/2 < \varepsilon. \qquad \square$$

Problem 2. Let $\mathcal{C}$ be the middle-thirds Cantor set constructed in the textbook. Show that $\mathcal{C}$ is compact, uncountable, and a null set.

Solution. $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$, where each set C_n is the union of 2^n disjoint closed intervals of length 3^n. The intersection of closed sets is always closed, so $\mathcal{C}$ is closed. Moreover, $\mathcal{C} \subset [0,1]$, so it is bounded. Therefore, by the Heine-Borel theorem we conclude that $\mathcal{C}$ is compact.

The set $\mathcal{C}$ is covered by C_n, which is a union of disjoint intervals, so by the definition of m^* we have

$$m^*(\mathcal{C}) \leq \ell(C_n) = \frac{2^n}{3^n}.$$

This holds for all $n \in \mathbb{N}$, so if we take limits as $n \to \infty$, we find $m^*(\mathcal{C}) = 0$.

An alternative way to represent $\mathcal{C}$ is all numbers in $[0,1]$ that have a ternary expansion using only the digits 0 and 2. Suppose that $\mathcal{C}$ were countable, so there would be an enumeration $\mathcal{C} = \{x_1, x_2, \ldots\}$. We write the ternary expansion of each x_n:

$$x_n = 0.x_{n1}x_{n2}x_{n3}\ldots,$$

where $x_{nk} \in \{0, 2\}$ for each $k \in \mathbb{N}$. Now define a number

$$y = 0.b_1b_2b_3\ldots$$

as follows. If $x_{nn=0}$, then set $b_n = 2$ and if $x_{nn=2}$, set $b_n = 0$. The number y has to lie in the Cantor set. However, it is not equal to any of the numbers x_n (Why? Couldn't x_n have two different ternary representations?). This is a contradiction. $\square$

Problem 3.

(i) Show that if $A \subset \mathbb{R}$ is an arbitrary set and $B \subset \mathbb{R}$ is a null set then $m^*(A \setminus B) = m^*(A)$. Conversely, show that if $B \subset \mathbb{R}$ is a set with the property that $m^*(A \setminus B) = m^*(A)$ for all sets $A \subset \mathbb{R}$, then B is a null set.

(ii) Show that if $A_1, A_2 \subset \mathbb{R}$ and $m^*(A_1 \cap A_2) = m^*(A_1 \cup A_2)$, then $m^*(A_1) = m^*(A_2)$. Does the converse hold?

(iii) Show that if $A \subset \mathbb{R}$ is a bounded set then $m^*(A) < \infty$. Does the converse hold?

(iv) Show that if $A \subset \mathbb{R}$ has non-empty interior then $m^*(A) > 0$.

Solution.

(i) Note that $A \setminus B \subset A$, so $m^*(A \setminus B) \leq m^*(A)$. On the other hand, $A = (A \setminus B) \cup (B \cap A)$ so by the subadditivity of m^* and the fact that $B \cap A \subset B$ we have

$$m^*(A) \leq m^*(A \setminus B) + m^*(B \cap A) \leq m^*(A \setminus B) + m^*(B) = m^*(A \setminus B).$$

For the converse, one can use $A = B$:

$$m^*(B) = m^*(B \setminus B) = m^*(\emptyset) = 0.$$

(ii) $A_1 \cap A_2 \subset A_1 \subset A_1 \cup A_2$. Hence,

$$m^*(A_1 \cap A_2) \leq m^*(A_1) \leq m^*(A_1 \cup A_2).$$

Since $m^*(A_1 \cap A_2) = m^*(A_1 \cup A_2)$, we have everywhere equality above, so $m^*(A_1) = m^*(A_1 \cap A_2)$. In the same way, $m^*(A_2) = m^*(A_1 \cap A_2)$, so we obtain the desired conclusion

The converse does not hold. For $A_1 = [0, 1]$ and $A_2 = [1, 2]$ we have $A_1 \cap A_2 = \{1\}$, so $m^*(A_1 \cap A_2) = \ell(\{1\}) = 0$ and $m^*(A_1 \cup A_2) = m^*([0, 2]) = \ell([0, 2]) = 2$.

(iii) Let A be bounded. Then there exists a bounded interval $I = (m, M)$ such that $A \subset I$; for example, one can take $m = \inf A$ and $M = \sup A$, which are both finite numbers. By the definition of m^* we have $m^*(A) \leq \ell(I) < \infty$.

The converse does not hold. Consider the set $A = \bigcup_{n=1}^{\infty}[n, n + 1/2^n]$, which is unbounded. However, by the subadditivity of m^* we have

$$m^*(A) \leq \sum_{n=1}^{\infty} m^*([n, n + 1/2^n]) = \sum_{n=1}^{\infty} 1/2^n = 1 < \infty.$$

(iv) Suppose that A has non-empty interior. This implies that there exists an open interval $I \subset A$ (why?). By monotonicity, we have

$$0 < m^*(I) \leq m^*(A)$$

so $m^*(A) > 0$. □

Problem 4. Let A be the subset of $(0, 1]$ consisting of all numbers whose *(non-terminating) base-*4 *expansion* (see Problem 5) does not have the digit 2. Find $m^*(A)$.
Hint: Note that the subset of $(0, 1]$ consisting of numbers whose first digit in their (non-terminating) base-4 expansion is different from 2 is $(0, 2/4] \cup [3/4, 1]$, so $A \subset (0, 2/4] \cup (3/4, 1]$ (why?). Hence, $m^*(A) \leq 1/2 + 1/4 = 3/4$ (why?). Using induction find a sequence r_n with $\lim_{n\to\infty} r_n = 0$ such that $m^*(A) \leq r_n$ for all $n \in \mathbb{N}$.

Solution. For each $n \in \mathbb{N}$ let A_n be the set of numbers of the form $0.a_1 \ldots a_n \ldots$ where $a_1, \ldots, a_n \in \{0, 1, 3\}$. We claim that the set A_n is the union of 3^n disjoint intervals of length $1/4^n$ and that $\ell(A_n) = 3^n/4^n$ for each $n \in \mathbb{N}$. The set A_n is the union of intervals of the form $(0.a_1 \ldots a_{n-1}0, 0.a_1 \ldots a_{n-1}1]$, $(0.a_1 \ldots a_{n-1}1, 0.a_1 \ldots a_{n-1}2]$, and $(0.a_1 \ldots a_{n-1}2, 0.a_1 \ldots a_{n-1}3]$. Each of these intervals has length $1/4^n$ and the total number of them (as $a_1 \ldots, a_{n-1}$ range over $0, 1, 3$) is 4^n.

Since $A \subset A_n$, by the definition of m^* we have

$$m^*(A) \leq \ell(A_n) = 3^n/4^n.$$

Letting $n \to \infty$ we obtain $m^*(A) = 0$.
Remark: note the similarity between this construction and the construction of the Cantor set. □

Problem 5 (Optional)**.** Show that every number in $(0,1]$ has a unique non-terminating *base-p expansion*, where p is a positive integer. In other words, for each $x \in (0,1]$ show that there exist unique numbers $k_n \in \{0,1,\dots,p\}$, $n \in \mathbb{N}$, such that

$$x = \sum_{n=1}^{\infty} \frac{k_n}{p^n}$$

and such that k_n is non-zero for infinitely many $n \in \mathbb{N}$.
Remark: The above equality can also be expressed as $x = 0.k_1k_2k_3\dots$ in base p. Which number does $0.111\dots$ in binary expansion (i.e, $p = 2$) represent?
Hint: Let k_1 be the largest integer such that $k_1/p < x$ (note that $0 \leq k_1 < p$); then let k_2 be the largest integer such that $k_1/p + k_2/p^2 < x$, and proceed inductively. Show then that with this definition of k_n we have $x = \sum_{n=1}^{\infty} k_n/p^n$.

Problem 6 (Optional)**.** For each $n \in \mathbb{N}$ consider a sequence $\{a_{nk}\}_{k\geq 1}$ of non-negative real numbers. Explain why the double sum

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{nk}$$

always converges and it is a non-negative number, possibly infinite. Let $\{b_j\}_{j\geq 1}$ be a rearrangement of $\{a_{nk}\}_{k,n\geq 1}$; that is, for each $j \in \mathbb{N}$ there exists a unique pair $(n,k) \in \mathbb{N}\times\mathbb{N}$ such that $b_j = a_{nk}$ and conversely for each pair $(n,k) \in \mathbb{N}\times\mathbb{N}$ there exists a unique $j \in \mathbb{N}$ such that $a_{nk} = b_j$. Show that

$$\sum_{j=1}^{\infty} b_j = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{nk}.$$

In particular, any rearrangement of $\{a_{nk}\}_{n,k\geq 1}$ gives the same sum. Does the same hold if the numbers a_{nk} are not assumed to be non-negative, even assuming that all series converge?
Hint: Think of an example where a_{nk} is $-1, 0$, or 1.

Problem 7 (Optional)**.** Show that the Cantor staircase function $f\colon [0,1] \to [0,1]$, as defined in the lecture, is continuous, increasing, satisfies $f(0) = 0$, $f(1) = 1$, and it is constant in each interval lying in the complement of the middle-thirds Cantor set. Therefore, all the increase of the function f occurs in a "negligible" set, the Cantor set, which is a null set.

Problem 8 (Optional)**.** Let $[a, b] \subset \mathbb{R}$ be a bounded interval, and suppose that $J_1, \ldots, J_m$ are open intervals whose union covers $[a, b]$. Show that

$$\ell([a, b]) = b - a \leq \sum_{i=1}^{m} \ell(J_i).$$