Homework 1 (due 09/05)

MAT 324: Real Analysis

Problem 1. Let $A \subset \mathbb{R}$ be a countable set. Show that A is a null set by proving that for each $\varepsilon > 0$ there exist *open* intervals I_n, $n \in \mathbb{N}$, such that $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$.

Problem 2. Let $\mathcal{C}$ be the middle-thirds Cantor set constructed in the textbook. Show that $\mathcal{C}$ is compact, uncountable, and a null set.

Problem 3.

(i) Show that if $A \subset \mathbb{R}$ is an arbitrary set and $B \subset \mathbb{R}$ is a null set then $m^*(A \setminus B) = m^*(A)$. Conversely, show that if $B \subset \mathbb{R}$ is a set with the property that $m^*(A \setminus B) = m^*(A)$ for all sets $A \subset \mathbb{R}$, then B is a null set.

(ii) Show that if $A_1, A_2 \subset \mathbb{R}$ and $m^*(A_1 \cap A_2) = m^*(A_1 \cup A_2)$, then $m^*(A_1) = m^*(A_2)$. Does the converse hold?

(iii) Show that if $A \subset \mathbb{R}$ is a bounded set then $m^*(A) < \infty$. Does the converse hold?

(iv) Show that if $A \subset \mathbb{R}$ has non-empty interior then $m^*(A) > 0$.

Problem 4. Let A be the subset of $(0, 1]$ consisting of all numbers whose *(non-terminating) base-*4 *expansion* (see Problem 5) does not have the digit 2. Find $m^*(A)$.
Hint: Note that the subset of $(0, 1]$ consisting of numbers whose first digit in their (non-terminating) base-4 expansion is different from 2 is $(0, 2/4] \cup [3/4, 1]$, so $A \subset (0, 2/4] \cup (3/4, 1]$ (why?). Hence, $m^*(A) \leq 1/2 + 1/4 = 3/4$ (why?). Using induction find a sequence r_n with $\lim_{n\to\infty} r_n = 0$ such that $m^*(A) \leq r_n$ for all $n \in \mathbb{N}$.

Problem 5 (Optional)**.** Show that every number in $(0,1]$ has a unique non-terminating *base-p expansion*, where p is a positive integer. In other words, for each $x \in (0,1]$ show that there exist unique numbers $k_n \in \{0,1,\dots,p\}$, $n \in \mathbb{N}$, such that

$$x = \sum_{n=1}^{\infty} \frac{k_n}{p^n}$$

and such that k_n is non-zero for infinitely many $n \in \mathbb{N}$.
Remark: The above equality can also be expressed as $x = 0.k_1k_2k_3\dots$ in base p. Which number does $0.111\dots$ in binary expansion (i.e, $p = 2$) represent?
Hint: Let k_1 be the largest integer such that $k_1/p < x$ (note that $0 \leq k_1 < p$); then let k_2 be the largest integer such that $k_1/p + k_2/p^2 < x$, and proceed inductively. Show then that with this definition of k_n we have $x = \sum_{n=1}^{\infty} k_n/p^n$.

Problem 6 (Optional)**.** For each $n \in \mathbb{N}$ consider a sequence $\{a_{nk}\}_{k\geq 1}$ of non-negative real numbers. Explain why the double sum

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{nk}$$

always converges and it is a non-negative number, possibly infinite. Let $\{b_j\}_{j\geq 1}$ be a rearrangement of $\{a_{nk}\}_{k,n\geq 1}$; that is, for each $j \in \mathbb{N}$ there exists a unique pair $(n,k) \in \mathbb{N}\times\mathbb{N}$ such that $b_j = a_{nk}$ and conversely for each pair $(n,k) \in \mathbb{N}\times\mathbb{N}$ there exists a unique $j \in \mathbb{N}$ such that $a_{nk} = b_j$. Show that

$$\sum_{j=1}^{\infty} b_j = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{nk}.$$

In particular, any rearrangement of $\{a_{nk}\}_{n,k\geq 1}$ gives the same sum. Does the same hold if the numbers a_{nk} are not assumed to be non-negative, even assuming that all series converge?
Hint: Think of an example where a_{nk} is $-1, 0$, or 1.

Problem 7 (Optional)**.** Show that the Cantor staircase function $f\colon [0,1] \to [0,1]$, as defined in the lecture, is continuous, increasing, satisfies $f(0) = 0$, $f(1) = 1$, and it is constant in each interval lying in the complement of the middle-thirds Cantor set. Therefore, all the increase of the function f occurs in a “negligible” set, the Cantor set, which is a null set.

Problem 8 (Optional)**.** Let $[a, b] \subset \mathbb{R}$ be a bounded interval, and suppose that $J_1, \ldots, J_m$ are open intervals whose union covers $[a, b]$. Show that

$$\ell([a, b]) = b - a \leq \sum_{i=1}^{m} \ell(J_i).$$