MAT 324: Real Analysis, Fall 2017
Final Exam (150 mins)

Name:

There are 9 problems on this exam of varying difficulty: Problem 1 below, Problem 2 on the back of this cover sheet, and Problems 3-9 on a separate sheet. Please write your *answers* to Problems 1 and 2 in the space provided on this cover sheet and *solutions* to Problems 3-9 on the provided printer paper, starting each problem on a new page and clearly indicating the number (and part) of the problem you are solving. You do *not* need to copy the statements of the problems. You can cite established statements from the textbook as appropriate (e.g. not statements you are being asked to establish).

Problem 1 (10pts)

List all σ-fields on the set $X \equiv \{a, b, c\}$ of three elements below, *one per line*. List *only*. Do *not* write any explanations or anything else. Penalty for missing σ-fields and for anything written below which is not a σ-field on X.

Problem 2 (10pts)

Determine whether each of the following statements is *true* (i.e. *always* holds) or *false* (otherwise). Clearly *circle O* the letters (a)-(j) corresponding to the true statements and *cross X* the letters corresponding to the false statements. Circle or cross the letters *only*. Do *not* write any explanations or anything else. Do *not* skip any of the questions, as there is no penalty for wrong answers on this problem.

(a) Let $(X,\mathcal{F},\mu)$ be a measure space. If $E_1,E_2,\ldots\in\mathcal{F}$ and $E_1\subset E_2\subset\ldots$, then

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{F} \qquad \text{and} \qquad \mu\bigg(\bigcup_{n=1}^{\infty} E_n\bigg) = \lim_{n\longrightarrow\infty}\mu(E_n).$$

(b) Let $(X,\mathcal{F},\mu)$ be a measure space. If $E_1,E_2,\ldots\in\mathcal{F}$ and $E_1\supset E_2\supset\ldots$, then

$$\bigcap_{n=1}^{\infty} E_n \in \mathcal{F} \qquad \text{and} \qquad \mu\bigg(\bigcap_{n=1}^{\infty} E_n\bigg) = \lim_{n\longrightarrow\infty}\mu(E_n).$$

(c) Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a map. If $f^{-1}(E)\subset\mathbb{R}$ is a Lebesgue measurable subset for every Lebesgue measurable subset $E\subset\mathbb{R}$, then f is a Lebesgue measurable function.

(d) Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a map. If f is a Lebesgue measurable function, then $f^{-1}(E)\subset\mathbb{R}$ is a Lebesgue measurable subset for every Lebesgue measurable subset $E\subset\mathbb{R}$.

(e) Let $(X,\mathcal{F},\mu)$ be a measure space and $f_n: X\longrightarrow\mathbb{R}^+$ be a sequence of measurable functions. If f_n converges pointwise to a function $f: X\longrightarrow\mathbb{R}$, then f is measurable and

$$\int_X f\,\mathrm{d}\mu = \lim_{n\longrightarrow\infty}\bigg(\int_X f_n\,\mathrm{d}\mu\bigg).$$

(f) Let $f_1,f_2,\ldots\in\mathcal{L}^1(\mathbb{R})$ be a sequence of continuous functions and $f\in\mathcal{L}^1(\mathbb{R})$ be another continuous function. If $f_n\longrightarrow f$ pointwise a.e. on $\mathbb{R}$, then there exists a subsequence f_{n_i} of f_n so that $f_{n_i}\longrightarrow f$ in the L^1-norm.

(g) Let $f_1,f_2,\ldots\in\mathcal{L}^1(\mathbb{R})$ be a sequence of continuous functions and $f\in\mathcal{L}^1(\mathbb{R})$ be another continuous function. If $f_n\longrightarrow f$ in the L^1-norm, then there exists a subsequence f_{n_i} of f_n so that $f_{n_i}\longrightarrow f$ pointwise a.e. on $\mathbb{R}$.

(h) $L^1([0,1])\subset L^2([0,1])$

(i) $L^2([0,1])\subset L^1([0,1])$

(j) If $f:[0,1]\times[2,3]\longrightarrow\mathbb{R}^+$ is Lebesgue measurable, then

$$\int_{[0,1]}\bigg(\int_{[2,3]} f(x,y)\mathrm{d}m(y)\bigg)\mathrm{d}m(x) = \int_{[2,3]}\bigg(\int_{[0,1]} f(x,y)\mathrm{d}m(x)\bigg)\mathrm{d}m(y)\,.$$

Problem 3 (10pts)

Let $(X,\mathcal{F},\mu)$ be a measure space.

(a) Give a definition of what this means.

(b) Show *directly* from the definition that $\mu(A\cup B) \le \mu(A)+\mu(B)$ for all $A,B\in\mathcal{F}$.

Problem 4 (15pts)

Let $A\subset\mathbb{R}$ be a null subset (with respect to the Lebesgue outer measure m^* on $\mathbb{R}$) and $B\subset\mathbb{R}$ be any subset. Show that

(a) $A\times B\subset\mathbb{R}^2$ is a null subset (with respect to the Lebesgue outer measure m_2^* on $\mathbb{R}^2$) if B is bounded (i.e. $B\subset[-R,R]$ for some $R\in\mathbb{R}^+$).

(b) $A\times B\subset\mathbb{R}^2$ is a null subset (whether or not B is bounded).

Note: For the purposes of (b), you can assume (a) even if you do not know how to do (a).

Problem 5 (15pts)

Let $f\colon\mathbb{R}\longrightarrow\mathbb{R}$ be a Lebesgue measurable function. For each $n\in\mathbb{Z}^+$, define

$$E_n = \left\{x\in\mathbb{R}\colon |f(x)|\le\frac{1}{n}\right\}.$$

(a) Explain why each set E_n is measurable. Is the sequence $E_1,E_2,\ldots$ of sets increasing or decreasing?

(b) Suppose in addition that f is integrable. Find $\displaystyle\lim_{n\longrightarrow\infty}\int_{E_n} f\,\mathrm{d}m$.

(c) Give an example showing that the last conclusion can fail if f is not assumed to be integrable.

Problem 6 (20pts)

Let $(X,\mathcal{F},\mu)$ be a measure space, $f_1,f_2,\ldots\colon X\longrightarrow\mathbb{R}$ be a sequence of functions, and $f\colon X\longrightarrow\mathbb{R}$ be another function.

(a) Define what it means for the sequence $f_1,f_2,\ldots$ to converge to f uniformly.

(b) Define what it means for the sequence $f_1,f_2,\ldots$ to converge to f in $L^\infty(X)$ or with respect to the $\|\cdot\|_\infty$-norm (this is *not* the same as uniformly; define the $\|\cdot\|_\infty$-norm if you refer to it).

(c) Show that the sequence $f_1,f_2,\ldots$ converges to f in $L^\infty(X)$ if and only if there exists $X_0\in\mathcal{F}$ such that $\mu(X_0)=0$ and the sequence $f_1|_{X-X_0},f_2|_{X-X_0},\ldots$ converges to $f|_{X-X_0}$ uniformly.

Problem 7 (20pts)

Let $(X,\mathcal{F},\mu)$ be a measure space and $f\colon X\longrightarrow\mathbb{R}^+$ be a measurable function.

(a) Suppose $f\in\mathcal{L}^1(X)$ and $\operatorname{ess\,sup} f<\infty$. Show that $f\in\mathcal{L}^3(X)$.

(b) Suppose $f\in\mathcal{L}^5(X)$ and $\operatorname{ess\,inf} f>0$. Show that $f\in\mathcal{L}^3(X)$.

(c) Suppose $f\in\mathcal{L}^1(X)\cap\mathcal{L}^5(X)$. Show that $f\in\mathcal{L}^3(X)$.

Hint for (c): break $f>0$ into two pieces to which (a) and (b) apply.

Problem 8 (25pts)

Let $(X,\mathcal{F},\mu)$ be a measure space and $g\colon X\longrightarrow\mathbb{R}^{\ge 0}$ be an $\mathcal{F}$-measurable function.

(a) Show that the map

$$\mu_g\colon \mathcal{F}\longrightarrow\overline{\mathbb{R}},\qquad \mu_g(E)=\int_E g\,\mathrm{d}\mu,$$

is a measure on $(X,\mathcal{F})$.

(b) Suppose $\mu(g^{-1}(0))=0$. Show that the measure space $(X,\mathcal{F},\mu)$ is complete if and only if the measure space $(X,\mathcal{F},\mu_g)$ is complete. Which implication can fail without the assumption $\mu(g^{-1}(0))=0$? Give an example.

(c) Show that

$$\int_E f\,\mathrm{d}\mu_g=\int_E fg\,\mathrm{d}\mu$$

for every $\mathcal{F}$-measurable function $f\colon X\longrightarrow\mathbb{R}^{\ge 0}$.

(d) Show that the above equality also holds for every μ-integrable function $f: X\longrightarrow\mathbb{C}$ if g is bounded. What can go wrong if g is not assumed to be bounded? Give an example.

Problem 9 (25pts)

For every $f\colon\mathbb{R}\longrightarrow\mathbb{R}$, define

$$\widehat{f}\colon\mathbb{R}\longrightarrow\mathbb{R},\qquad \widehat{f}(x)=f(x-1)\quad\forall\,x\in\mathbb{R}.$$

(a) Let $p\in[1,\infty]$. Show that

$$\Phi\colon L^p(\mathbb{R})\longrightarrow L^p(\mathbb{R}),\qquad \Phi\big([f]\big)=\big[\widehat{f}\big],$$

is a well-defined isometry and is a homomorphism of vector spaces over $\mathbb{R}$ (there are at least three things to show here: *well-defined*, *isometry*, and *homomorphism*).

(b) Suppose $[f]\in L^\infty(\mathbb{R})$ is nonzero and $\Phi[f]=c[f]$ for some $c\in\mathbb{R}$. Show that $c=\pm1$ and that both cases are possible (find f that works for $c=1$ and another f that works for $c=-1$).

(c) Suppose $p\in[1,\infty)$, $[f]\in L^p(\mathbb{R})$, and $\Phi[f]=c[f]$ for some $c\in\mathbb{R}$. Show that $[f]=0$.