MAT 324 – Real Analysis

Fall 2014

Final Exam (Solutions) – December 12, 2014

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator or an electronic device.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

Problem 1: (20 points) Determine whether the following statements are true or false. No further explanation is necessary.

(1) TRUE

The outer measure is countably subadditive.

(2) FALSE

All Lebesgue measurable sets are also Borel sets.

(3) FALSE

For any subset $A \subset [0,1]$ the characteristic function $\mathbb{1}_A$ defined by $\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$ is measurable.

(4) TRUE

$f \in L^1(\mathbb{R})$ if and only if $|f| \in L^1(\mathbb{R})$.

(5) FALSE

It is possible to define an inner product on the space $L^4[0,1]$ with the norm $\|\cdot\|_4$.

(6) FALSE

There exist functions which belong to $L^4(0,1)$ but not to $L^2(0,1)$.

(7) TRUE

If $f, g \in L^1(\mathbb{R})$ then $h \in L^1(\mathbb{R}^2)$, where $h(x,y) = f(x)g(y)$.

(8) TRUE

Let $\mu(E) = \displaystyle\int_E e^{-\pi x^2} dx$ for any Borel subset $E \subset \mathbb{R}$. Then $\mu(\mathbb{R}) = 1$ and $\mu \ll m$.

(9) TRUE

Let μ, ν and λ be σ-finite measures. If $\mu \ll \nu$ and $\nu \ll \lambda$ then $\mu \ll \lambda$.

(10) FALSE

Let $\lambda_1, \lambda_2, \mu$ be σ-finite measures on a σ-field $\mathcal{F}$. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then $\lambda_1 \perp \lambda_2$.

Problem 2: (12 points) Consider the set $K = \{\frac{1}{n} \mid n \geq 1\}$ and the function

$$f(x) = \begin{cases} x^2 & \text{if } x \notin K \\ 1 & \text{if } x \in K. \end{cases}$$

Explain why f is measurable and compute $\displaystyle\int_{[0,1]-K} f\, dm$.

Solution. The set K is countable so $m(K) = 0$. The function $f = x^2$ a.e. and $x \to x^2$ is a continuous function, hence measurable. This implies that f is measurable and $\int_{[0,1]-K} f\, dm = \int_0^1 x^2\, dx = \frac{1}{3}$. □

Problem 3: (10 points) Let $E \subset [0,1]$ be a measurable set such that for any interval $(a,b) \subset [0,1]$ we have

$$m(E \cap (a,b)) \geq \frac{1}{2}(b-a).$$

Show that $m(E) = 1$.

SOLUTION. We look at the complement of E and note that

$$m(E^c \cap (a,b)) \leq \frac{1}{2}(b-a).$$

It follows (an shown in class) that $m(E^c) = 0$ so $m(E) = 1$. □

Problem 4: (12 points) Let $f \in L^1(0,1)$. Compute the following limit if it exists or explain why it does not exist:

$$\lim_{k\to\infty} \int_0^1 k \ln\left(1 + \frac{|f(x)|}{k^2}\right) dx.$$

Solution. Note that

$$\left(1 + \frac{|f(x)|}{k^2}\right)^{k^2}$$

is an increasing function of k and increases to $e^{|f(x)|}$. Therefore

$$k \ln\left(1 + \frac{|f(x)|}{k^2}\right) = \frac{1}{k}\left(1 + \frac{|f(x)|}{k^2}\right)^{k^2} \leq \frac{|f(x)|}{k} \leq |f(x)|.$$

Since $f \in L^1(0,1)$, we can apply DCT and get that

$$\lim_{k\to\infty} \int_0^1 k \ln\left(1 + \frac{|f(x)|}{k^2}\right) dx \leq \lim_{k\to\infty} \int_0^1 \frac{|f(x)|}{k}\, dx = \lim_{k\to\infty} \frac{\|f\|_1}{k} = 0.$$

□

Problem 5: (12 points) Compute

$$\int_{(0,\pi)\times(0,\infty)} xy\sin(x)e^{-xy^2}\,dxdy$$

and explain how Fubini's theorem is used.

Solution. By Tonelli-Fubini Theorem, we get that

$$\begin{aligned}\int_{(0,\pi)\times(0,\infty)} xy\sin(x)e^{-xy^2}\,dxdy &= \int_0^\pi\int_0^\infty xy\sin(x)e^{-xy^2}\,dydx = \int_0^\pi \sin(x)\frac{1}{2}e^{-xy^2}\Big|_0^\infty\,dx \\ &= -\frac{1}{2}\cos(x)\Big|_0^\pi = 1.\end{aligned}$$

□

Problem 6: (12 points) Consider the measurable space $([0,1], \mathcal{F})$, where $\mathcal{F} = \mathcal{B}_{[0,1]}$ is the σ-algebra of Borel subsets of $[0,1]$. Let μ be a σ-finite measure on $\mathcal{F}$ with $\mu \ll m$. Let ν be the counting measure on $\mathcal{F}$, that is $\nu(E) =$ number of elements in E if E is finite and $\nu(E) = \infty$ otherwise. Let C be the Cantor middle-thirds set from $[0,1]$.

a) Explain why $D = C \times C$ belongs to the product σ-algebra $\mathcal{F} \times \mathcal{F}$.

Solution. We know that $C \in \mathcal{F}$ so, by definition of $\mathcal{F} \times \mathcal{F}$, $D \in \mathcal{F} \times \mathcal{F}$. $\square$

b) Compute $\int_0^1 \int_0^1 \mathbb{1}_D(x,y)\, d\mu(x) d\nu(y)$. Recall that $\mathbb{1}_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D. \end{cases}$

Solution. The value of the integral is 0. $\square$

c) Compute $\int_0^1 \int_0^1 \mathbb{1}_D(x,y) d\nu(y) d\mu(x)$. Does this example contradict Fubini's theorem?

Solution. The value of the integral is also 0. The integrals are the same, so this does not contradict Fubini's theorem. However Fubini's theorem does not apply here because ν is not a σ-finite measure. $\square$

Problem 7: (12 points) Suppose μ is a σ-finite measure on $([0,1],\mathcal{F})$ and E_1, E_2 are two measurable subsets of $[0,1]$. Define ν by

$$\nu(E) = \frac{1}{4}\mu(E \cap E_1) + \frac{3}{4}\mu(E \cap E_2), \text{ for all } \quad E \in \mathcal{F}.$$

a) Compute $\nu(E_1 \cap E_2)$.

SOLUTION. $\nu(E_1 \cap E_2) = \mu(E_1 \cap E_2)$. □

b) Show that $\nu \ll \mu$.

SOLUTION. Clearly if $\mu(E) = 0$ then $\mu(E \cap E_1) = 0$ and $\mu(E \cap E_2) = 0$. So $\nu(E) = 0$ and $\nu \ll \mu$. □

c) Find the Radon-Nikodym derivative $\dfrac{d\nu}{d\mu}$.

SOLUTION. As in the homework, the Radon-Nikodym derivative is $\frac{1}{4}\mathbb{1}_{E_1} + \frac{3}{4}\mathbb{1}_{E_2}$. □

Problem 8: (10 points) Let $f \in L^2(0,\infty) \cap L^5(0,\infty)$. Show that $f \in L^3(0,\infty)$.

SOLUTION. Define sets $E_1 = \{x \in (0,\infty) \mid |f(x)| \le 1\}$ and $E_2 = \{x \in (0,\infty) \mid |f(x)| > 1\}$. Then E_1 and E_2 are disjoint and $E_1 \cup E_2 = (0,\infty)$. We have

$$\begin{aligned}\int_0^\infty |f(x)|^3\,dx &= \int_{E_1} |f(x)|^3\,dx + \int_{E_2} |f(x)|^3\,dx \\ &\le \int_{E_1} |f(x)|^2\,dx + \int_{E_2} |f(x)|^5\,dx < \infty.\end{aligned}$$

So $f \in L^3(0,\infty)$. $\square$