

MAT 324 – Real Analysis
FALL 2014

Final Exam – December 12, 2014

NAME: _____

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator or an electronic device.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

Problem 1: (20 points) Determine whether the following statements are true or false. No further explanation is necessary.

(1) TRUE FALSE

The outer measure is countably subadditive.

(2) TRUE FALSE

All Lebesgue measurable sets are also Borel sets.

(3) TRUE FALSE

For any subset $A \subset [0, 1]$ the characteristic function $\mathbf{1}_A$ defined by $\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$ is measurable.

(4) TRUE FALSE

$f \in L^1(\mathbb{R})$ if and only if $|f| \in L^1(\mathbb{R})$.

(5) TRUE FALSE

It is possible to define an inner product on the space $L^4[0, 1]$ with the norm $\|\cdot\|_4$.

(6) TRUE FALSE

There exist functions which belong to $L^4(0, 1)$ but not to $L^2(0, 1)$.

(7) TRUE FALSE

If $f, g \in L^1(\mathbb{R})$ then $h \in L^1(\mathbb{R}^2)$, where $h(x, y) = f(x)g(y)$.

(8) TRUE FALSE

Let $\mu(E) = \int_E e^{-\pi x^2} dx$ for any Borel subset $E \subset \mathbb{R}$. Then $\mu(\mathbb{R}) = 1$ and $\mu \ll m$.

(9) TRUE FALSE

Let μ, ν and λ be σ -finite measures. If $\mu \ll \nu$ and $\nu \ll \lambda$ then $\mu \ll \lambda$.

(10) TRUE FALSE

Let $\lambda_1, \lambda_2, \mu$ be σ -finite measures on a σ -field \mathcal{F} . If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then $\lambda_1 \perp \lambda_2$.

Problem 2: (12 points) Consider the set $K = \{\frac{1}{n} \mid n \geq 1\}$ and the function

$$f(x) = \begin{cases} x^2 & \text{if } x \notin K \\ 1 & \text{if } x \in K. \end{cases}$$

Explain why f is measurable and compute $\int_{[0,1]-K} f \, dm$.

Problem 3: (10 points) Let $E \subset [0, 1]$ be a measurable set such that for any interval $(a, b) \subset [0, 1]$ we have

$$m(E \cap (a, b)) \geq \frac{1}{2}(b - a).$$

Show that $m(E) = 1$.

Problem 4: (12 points) Let $f \in L^1(0, 1)$. Compute the following limit if it exists or explain why it does not exist:

$$\lim_{k \rightarrow \infty} \int_0^1 k \ln \left(1 + \frac{|f(x)|}{k^2} \right) dx.$$

Problem 5: (12 points) Compute

$$\int_{(0,\pi) \times (0,\infty)} xy \sin(x) e^{-xy^2} dx dy$$

and explain how Fubini's theorem is used.

Problem 6: (12 points) Consider the measurable space $([0, 1], \mathcal{F})$, where $\mathcal{F} = \mathcal{B}_{[0,1]}$ is the σ -algebra of Borel subsets of $[0, 1]$. Let μ be a σ -finite measure on \mathcal{F} with $\mu \ll m$. Let ν be the counting measure on \mathcal{F} , that is $\nu(E) = \text{number of elements in } E$ if E is finite and $\nu(E) = \infty$ otherwise. Let C be the Cantor middle-thirds set from $[0, 1]$.

a) Explain why $D = C \times C$ belongs to the product σ -algebra $\mathcal{F} \times \mathcal{F}$.

b) Compute $\int_0^1 \int_0^1 \mathbb{1}_D(x, y) d\mu(x) d\nu(y)$. Recall that $\mathbb{1}_D(x, y) = \begin{cases} 1 & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \notin D. \end{cases}$

c) Compute $\int_0^1 \int_0^1 \mathbb{1}_D(x, y) d\nu(y) d\mu(x)$. Does this example contradict Fubini's theorem?

Problem 7: (12 points) Suppose μ is a σ -finite measure on $([0, 1], \mathcal{F})$ and E_1, E_2 are two measurable subsets of $[0, 1]$. Define ν by

$$\nu(E) = \frac{1}{4}\mu(E \cap E_1) + \frac{3}{4}\mu(E \cap E_2), \text{ for all } E \in \mathcal{F}.$$

a) Compute $\nu(E_1 \cap E_2)$.

b) Show that $\nu \ll \mu$.

c) Find the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Problem 8: (10 points) Let $f \in L^2(0, \infty) \cap L^5(0, \infty)$. Show that $f \in L^3(0, \infty)$.