MAT 324 – Real Analysis
Fall 2016

Midterm – October 25, 2016
Solutions

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	**SCORE**
1	
2	
3	
4	
5	
TOTAL	

Problem 1: (21 points) Let E a null subset of $\mathbb{R}$.

a) Use the definition of a null set to show that the set $-E = \{-x \mid x \in E\}$ is null.

SOLUTION. The set E is null, so for every $\epsilon > 0$ there exists a cover $E \subset \cup_{j=1}^{\infty} I_j$, of open intervals I_j, such that $\sum\limits_{j=1}^{\infty} m(I_j) < \epsilon$. Note that $\bigcup_{j=1}^{\infty}(-I_j)$ is a cover for the set $-E$. This follows directly from the definition of the set $-E$. One can easily check that $m(I_j) = m(-I_j)$, so $\sum\limits_{j=1}^{\infty} m(-I_j) < \epsilon$. This shows that $-E$ is null. $\square$

b) Consider $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. Is $f^{-1}(E)$ measurable? Explain.

SOLUTION. Note that f is bijective and $f^{-1}(x) = x^2$, $f^{-1} : [0, \infty) \to [0, \infty)$. The function f is continuous, so it is measurable. The set E is null, but it may not be Borel (note that not all null sets are Borel!). So we cannot draw the conclusion that $f^{-1}(E)$ is measurable from these observations. However, by definition,

$$f^{-1}(E) = \{x^2 \mid x \in E, x \geq 0\},$$

so $f^{-1}(E) \subset F$, where F is defined in part c). By part c) F is null, which implies that $f^{-1}(E)$ is null and hence Lebesgue measurable. We are left to prove part c).

One could also prove directly that $f^{-1}(E)$ is null using the definition of a null set as in part a); the proof would be exactly the same as in part c) below. $\square$

c) Let $F = \{x^2 \mid x \in E\}$. Show that F is null.

SOLUTION. Suppose $F \subset [-N, N]$ for some large N, say $N > 100$. Observe that $|x| \leq N$ for all $x \in F$. Let $\epsilon > 0$ be small enough, say $\epsilon < N/2$. Let $E \subset \bigcup_{j=1}^{\infty} I_j$ be an open cover of the null set E, as in part a), such that $\sum\limits_{j=1}^{\infty} m(I_j) < \epsilon$. Then $F \subset \bigcup_{j=1}^{\infty} I_j^2$. If $I_j = (a_j, b_j)$, then $I_j^2 = (a_j^2, b_j^2)$ or $I_j^2 = (b_j^2, a_j^2)$, depending whether a_j, b_j are positive or negative. In any case I_j^2 is an interval. Suppose without loss of generality that $0 < a_j < b_j$ (the other cases are treated similarly). Then

$$m(I_j^2) = b_j^2 - a_j^2 = (b_j - a_j)(b_j + a_j) < 4N m(I_j).$$

Note that if I_j is an interval that is part of the cover for E, then $I_j \subset [-\sqrt{N} - \epsilon, \sqrt{N} + \epsilon]$. So $|a_j|$ and $|b_j|$ are $\leq 2\sqrt{N} + 2\epsilon < 2N$, which justifies the inequality above. These inequalities are not optimal and other similar bounds work. It follows that

$$\sum_{j=1}^{\infty} m(I_j^2) < 4N \sum_{j=1}^{\infty} m(I_j) < 4N\epsilon,$$

which can be made arbitrarily small. This shows that F is null.

If F is unbounded then we write $F = \bigcup_{N=100}^{\infty} (F \cap [-N, N])$. This is an increasing union of bounded null sets, so it is null. $\square$

Problem 2: (20 points) Does there exist a Lebesgue measurable set $E \subset \mathbb{R}$ such that

$$m(E \cap I) \geq 0.99 \cdot m(E^c \cap I),$$

for every interval I? Either give an example of such a set, or prove that it does not exist.

Solution. Any measurable set E with $m(E^c) = 0$ verifies this inequality. For example, one could take $E = \mathbb{R}$, for which $E^c = \emptyset$. We explain below why these examples arise naturally.

From the definition of a measurable set, we have $m(I) = m(E \cap I) + m(E^c \cap I)$ for every interval I. Using the inequality from the hypothesis, we find

$$m(I) = m(E \cap I) + m(E^c \cap I) \geq 1.99 m(E^c \cap I).$$

However, if $m(E^c) > 0$ then, we know from HW that for every constant $0 < \alpha < 1$ there exists an interval I such that $m(E^c \cap I) \geq \alpha m(I)$. In particular, for $\alpha = 0.99$, there exists an interval I such that $m(E^c \cap I) > 0.99 m(I)$. But then

$$m(I) \geq 1.99 m(E^c \cap I) > 1.99 \cdot 0.99 m(I) > 1.5 m(I),$$

which is a contradiction. So we are left with the case $m(E^c) = 0$, for which the inequality is satisfied. $\square$

Problem 3: (25 points) Let E be a measurable set with $m(E) < \infty$. Let $f : E \to [0,1]$ be an integrable nonnegative function. For each $n \geq 0$ define the sets

$$E_{2^n,j} = \left\{ x \in E \mid \frac{j}{2^n} < f(x) \leq \frac{j+1}{2^n} \right\}, \quad \text{for} \quad j = 0, 1, \ldots, 2^n - 1.$$

and the functions

$$f_n(x) = \sum_{j=0}^{2^n-1} \frac{j+1}{2^n} \chi_{E_{2^n,j}}(x).$$

a) Explain why f_n is a measurable function and why $f_n(x) \geq f(x)$ for all $x \in E$.

Solution. The function f is integrablem, so it is measurable. Therefore each set $E_{2^n,j}$ is measurable. The function f_n is a finite sum of measurable functions, so it is measurable (in fact, it is a simple function). By construction, on the set $E_{2^n,j}$, we have $f(x) \leq \frac{j+1}{2^n} = f_n(x)$. Also, for fixed n, the sets $E_{2^n,j}$, for $j = 0, 1, \ldots, 2^n - 1$, are pairwise disjoint. Let $E_0 = \{x \in E \mid f(x) = 0\}$. The $E - E_0 = \bigcup_{j=0}^{2^n-1} E_{2^n,j}$. On the set E_0 we have $f_n(x) = f(x) = 0$. In conclusion, $f_n(x) \geq f(x)$ for all $x \in E$. $\square$

b) Prove that $(f_n)_{n\geq 1}$ is decreasing, that is, $f_{n+1}(x) \leq f_n(x)$ pointwise for every $x \in E$.

Solution. From the observations in part a), it is enough to prove that $f_{n+1}(x) \leq f_n(x)$ for $x \in E_{2^n,j}$, for $j = 0, 1, \ldots, 2^n - 1$. Note that

$$E_{2^n,j} = E_{2^{n+1},2j} \cup E_{2^{n+1},2j+1}.$$

If $x \in E_{2^{n+1},2j}$ then $f_{n+1}(x) = \frac{2j+1}{2^{n+1}} = \frac{j+1/2}{2^n}$ and $f_n(x) = \frac{j+1}{2^n}$, from the definition of the functions f_n and f_{n+1}. We have $f_{n+1}(x) < f_n(x)$. If $x \in E_{2^{n+1},2j+1}$ then $f_{n+1}(x) = \frac{2j+2}{2^{n+1}} = \frac{j+1}{2^n} = f_n(x)$. It follows that $f_{n+1}(x) \leq f_n(x)$ for $x \in E$. $\square$

c) Prove that

$$\int_E f\,dm = \lim_{n\to\infty} \sum_{j=0}^{2^n-1} \frac{j+1}{2^n} m(E_{2^n,j}).$$

SOLUTION. First note that $(f_n)_{n\geq 1}$ is a decreasing sequence of nonnegative measurable functions. By construction of the function f_n, observe that $0 \leq f_n - f(x) \leq \frac{1}{n}$, so $f_n(x) \to f(x)$ as $n \to \infty$. The first term of the sequence is integrable. Indeed,

$$f_1(x) = \frac{1}{2}\chi_{E_{2,0}}(x) + \chi_{E_{2,1}}(x)$$

and $\int_E f_1\,dm = \frac{1}{2}m(E_{2,0}) + m(E_{2,1}) < 2m(E) < \infty$.
We can apply MCT (for a decreasing sequence, as in the HW) and obtain

$$\int_E f\,dm = \int_E \lim_{n\to\infty} f_n(x)\,dm = \lim_{n\to\infty}\int_E f_n\,dm = \lim_{n\to\infty} \sum_{j=0}^{2^n-1} \frac{j+1}{2^n} m(E_{2^n,j}).$$

The last equality is just the definition of the integral of a simple function. $\square$

Problem 4: (22 points) Compute the following limit if it exists and justify the calculations. If the limit does not exist explain why it does not exist.

a) $\displaystyle\lim_{n\to\infty}\int_0^{100}\left(1-\frac{n\cos x}{1+n^2\sqrt{x}}\right)dx$

SOLUTION. Note that $1+n^2\sqrt{x}>n\sqrt{x}$ for all $n\geq 1$. We have

$$\left|1-\frac{n\cos x}{1+n^2\sqrt{x}}\right|\leq 1+\frac{n}{1+n^2\sqrt{x}}<1+\frac{n}{n\sqrt{x}}=1+x^{-1/2}.$$

The function $1+x^{-1/2}$ is integrable on $[0,100]$ since

$$\int_0^{100}1+x^{-1/2}\,dx=(x+2x^{1/2})\Big|_0^{100}=120.$$

We can therefore apply DCT and get

$$\lim_{n\to\infty}\int_0^{100}\left(1-\frac{n\cos x}{1+n^2\sqrt{x}}\right)dx=\int_0^{100}\lim_{n\to\infty}\left(1-\frac{n\cos x}{1+n^2\sqrt{x}}\right)dx=100.$$

□

b) $\displaystyle\lim_{n\to\infty}\int_0^{100}\frac{ne^{-nx}}{1+x^n}\,dx$

SOLUTION. We make a change of variables $y=nx$ and get

$$\int_0^{100}\frac{ne^{-nx}}{1+x^n}\,dx=\int_0^{100n}\frac{e^{-y}}{1+\left(\frac{y}{n}\right)^n}\,dy=\int_0^{\infty}\frac{e^{-y}}{1+\left(\frac{y}{n}\right)^n}\chi_{[0,100n]}(y)\,dy.$$

The sequence of functions $f_n(y)=\dfrac{e^{-y}}{1+\left(\frac{y}{n}\right)^n}\chi_{[0,100n]}(y)$, $n\geq 1$, is bounded above by $g(y)=e^{-y}\chi_{[0,\infty)}$, which is nonnegative and Riemann integrable since

$$\int_0^{\infty}e^{-y}\,dy=-e^{-y}\Big|_0^{\infty}=1.$$

Hence g is also Lebesgue integrable and we can apply DCT. We have

$$\lim_{n\to\infty}\int_0^{\infty}f_n(y)\,dy=\int_0^{\infty}\lim_{n\to\infty}\frac{e^{-y}}{1+\left(\frac{y}{n}\right)^n}\,dy=\int_0^1 e^{-y}\,dy=1-e^{-1}.$$

We have use the fact that $\lim_{n\to\infty}y^n=\infty$ if $y>1$, but $\lim_{n\to\infty}y^n=0$, if $0\leq y<1$. Also, recall that $\lim_{n\to\infty}n^n=1$. This gives

$$\lim_{n\to\infty}\frac{e^{-y}}{1+\left(\frac{y}{n}\right)^n}=\begin{cases}0 & \text{if } y>1\\ e^{-y}/2 & \text{if } y=1\\ e^{-y} & \text{if } 0\leq y<1.\end{cases}$$

In conclusion, $\displaystyle\lim_{n\to\infty}\int_0^{100}\frac{ne^{-nx}}{1+x^n}\,dx=1-e^{-1}.$ □

Problem 5: (12 points) Let $f : [0,1] \to \mathbb{R}$ be a function which is continuous everywhere except on a null set $E \subset [\frac{1}{4}, \frac{3}{4}]$. Compute the following limit and justify the calculations:

$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx.$$

Solution. The function is continuous a.e. so it is Riemann integrable and bounded. Moreover, it is Lebesgue integrable, measurable, and the Riemann and Lebesgue integrals agree. Let $f_n(x) = f(x^n)$. Since f is bounded we have $|f_n(x)| < M$, for all $x \in [0,1]$ and all $n \geq 1$. Each f_n is continuous a.e. (as a composition of f, which is continuous a.e., and the continuous function x^n), so it is measurable. We can apply DCT (or the uniform boundedness principle) and conclude that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = \int_0^1 \lim_{n\to\infty} f(x^n)\, dx = f(0).$$

Note that f is continuous at $x = 0$.

The sequence $(x^n)_{n\geq 1}$ is decreasing on the interval $[0,1)$, but we don't know whether $f_n(x) = f(x^n)$ is monotone (either increasing or decreasing). Also we don't know whether f is nonnegative or not, so we cannot apply MCT directly. $\square$