MAT 324 – Real Analysis
Fall 2014

Midterm – October 23, 2014

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
TOTAL	

Problem 1: (25 points) Let $E_1, E_2, \dots, E_{2014}$ be measurable subsets of $[0,1]$.

a) Suppose $m(E_k) > 1 - \dfrac{1}{2^k}$ for each $1 \le k \le 2014$. Show that $m\left(\bigcap_{k=1}^{2014} E_k\right) > 0$.

Solution. Note that

$$m\left(\bigcup_{k=1}^{2014} E_k^c\right) \le \sum_{k=1}^{2014} m(E_k^c) = \sum_{k=1}^{2014}(1 - m(E_k)) = 2014 - \sum_{k=1}^{2014} m(E_k),$$

by subadditivity and taking complements. Therefore

$$m\left(\bigcap_{k=1}^{2014} E_k\right) = 1 - m\left(\bigcup_{k=1}^{2014} E_k^c\right) \ge 1 - 2014 + \sum_{k=1}^{2014} m(E_k) = \sum_{k=1}^{2014} m(E_k) - 2013 > \frac{1}{2^{2015}} > 0.$$

The last inequality follows from the fact that for each E_k we have $m(E_k) > 1 - \frac{1}{2^k}$ so

$$\sum_{k=1}^{2014} m(E_k) > \sum_{k=1}^{2014}\left(1 - \frac{1}{2^k}\right) = 2013 + \frac{1}{2^{2015}}. \qquad \square$$

b) Suppose almost every x from the interval $[0,1]$ belongs to at least 3 of these subsets. Prove that there exists at least one set E_k with $1 \le k \le 2014$ such that $m(E_k) \ge \frac{3}{2014}$.
Hint: The function $f(x) = \sum_{k=1}^{2014} \chi_{E_k}(x)$ has the property that $f(x) \ge 3$ a.e.

Solution. We follow the hint and set $f(x) = \sum_{k=1}^{2014} \chi_{E_k}(x)$. Since almost every x from the interval $[0,1]$ belongs to at least 3 sets E_k we have that $f(x) \ge 3$ almost everywhere. The function f is a simple function with finite support, hence it is measurable and integrable and $\int_{[0,1]} f\,dm = \sum_{k=1}^{2014} m(E_k)$. Suppose that $m(E_k) < \frac{3}{2014}$. Then

$$3 \le \int_{[0,1]} f\,dm = \sum_{k=1}^{2014} m(E_k) < 2014 \cdot \frac{3}{2014} = 3,$$

so $3 < 3$, which is a contradiction. Hence there exists at least one set E_k such that $m(E_k) \ge \frac{3}{2014}$. $\square$

Problem 2: (20 points) Does there exist a Lebesgue measurable subset E of $\mathbb{R}$ such that for every interval (a,b) we have

$$m\left(E \cap (a,b)\right) = \frac{b-a}{2} \,?$$

Either construct such a set or prove it does not exist.

Solution 1. Suppose such a set E exists. Then $E \cap (0,2)$ is a bounded measurable set with $m(E \cap (0,2)) = 1$. Let $0 < \epsilon < 1$. By the outer regularity property applied to $E \cap (0,2)$ there exists an open set O such that $E \cap (0,2) \subset O$ and $m(O - E \cap (0,2)) < \epsilon$. It follows that $m(O) < m(E \cap (0,2)) + \epsilon = 1 + \epsilon$. The set O is open so it can be written as a disjoint union of intervals $O = \bigcup_{k=1}^{\infty} I_k$, with $I_k = (a_k, b_k)$.

By hypothesis we have that $m(E \cap I_k) = \frac{m(I_k)}{2}$ for all $k \geq 1$. Hence

$$m(E \cap O) = m\left(E \cap \bigcup_{k=1}^{\infty} I_k\right) \leq \sum_{k=1}^{\infty} m(E \cap I_k) = \frac{1}{2}\sum_{k=1}^{\infty} m(I_k) = \frac{m(O)}{2} < \frac{1+\epsilon}{2}.$$

However $E \cap (0,2) \subset O$ so $E \cap (0,2) \subset E \cap O$ and $1 = m(E \cap (0,2)) \leq m(E \cap O) < \frac{1+\epsilon}{2}$. We have obtained that $1 < \frac{1+\epsilon}{2}$ which gives $1 < \epsilon$. Contradiction! So there is no set E which verifies the hypothesis. $\square$

Solution 2. This solution reduces the problem to the discussion from class. From the hypothesis we know that $m\left(E \cap (-n,n)\right) = \frac{n-(-n)}{2} = n \to \infty$ as $n \to \infty$. So the measure $m(E \cap \mathbb{R}) = m(E) = \infty$, hence E has to be unbounded.

However, $E_n = E \cap (-n,n)$ is bounded for every n. We claim that

$$m\left(E_n \cap (a,b)\right) \leq \frac{b-a}{2}$$

for every interval (a,b). There are four cases to consider, depending on how the interval (a,b) intersects the interval $(-n,n)$. If $(a,b) \subset (-n,n)$ then $E \cap (-n,n) \cap (a,b) = E \cap (a,b)$ and $m(E \cap (a,b)) = \frac{b-a}{2}$. If $(a,b) \cap (-n,n) = (n,b)$ (this is possible if $a < n < b$) then $m(E \cap (-n,n) \cap (a,b)) = m(E \cap (n,b)) = \frac{b-n}{2} \leq \frac{b-a}{2}$. Similarly if $(a,b) \cap (-n,n) = (a,-n)$ then $m(E \cap (-n,n) \cap (a,b)) = m(E \cap (a,-n)) = \frac{-n-a}{2} \leq \frac{b-a}{2}$ since $-n < b$. Finally, if $(-n,n) \subset (a,b)$ then $m(E \cap (-n,n) \cap (a,b)) = m(E \cap (-n,n)) = n \leq \frac{b-a}{2}$ since the length of (a,b) is greater than the length of $(-n,n)$, which is $2n$.

It now follows that $m(E_n) = 0$. Since n was arbitrary we get that E is null. Contradiction! So there is no such set E with this property. $\square$

Recall that in class we have shown the following (but the proof was not required):
Fact : Let $0 < a < 1$ and suppose E is a bounded measurable set such that

$$m(E \cap I) \leq am(I)$$

for every interval I. Then $m(E) = 0$. A particular case is $a = \frac{1}{2}$.

Problem 3: (25 points) Let E be a measurable set and $f : E \to \mathbb{R}$ Lebesgue integrable on E. Define $E_k = \left\{x \in E \mid |f(x)| < \frac{1}{k}\right\}$ for $k \geq 1$.

a) Show that each E_k is a measurable set.

SOLUTION. The function f is measurable (since it is integrable) and

$$E_k = f^{-1}((-\infty, \frac{1}{k})) \cap f^{-1}((-\frac{1}{k}, \infty))$$

is a measurable set. By definition, since f is measurable $f^{-1}((-\infty, \frac{1}{k}))$ and $f^{-1}((-\frac{1}{k}, \infty))$ are both measurable sets. $\square$

b) Determine whether $\{E_k\}$ is an increasing or decreasing collection of sets.

SOLUTION. Since $|f(x)| < \frac{1}{k+1}$ implies $|f(x)| < \frac{1}{k}$ we have that $E_{k+1} \subset E_k$, so $\{E_k\}$ is an increasing collection of sets. $\square$

c) Show that $\lim\limits_{k\to\infty} \int_{E_k} |f|\, dm = 0$.

SOLUTION. Let $f_k = |f| \cdot \chi_{E_k}$ for $k \geq 1$. Then f_k is a decreasing sequence of non-negative measurable functions (product of two measurable functions) from part b). Note that each characteristic function χ_{E_k} is measurable from part a). Also, if f is measurable, then $|f|$ is measurable. We have $\int_E |f|\, dm < \infty$ since f is integrable. Since $E_k \subset E$ this gives that $\int_{E_1} f_1\, dm < \infty$. By the Monotone Convergence Theorem (the decreasing version from homework),

$$\lim_{k\to\infty} \int_{E_k} |f|\, dm = \lim_{k\to\infty} \int_E |f| \cdot \chi_{E_k}\, dm = \lim_{k\to\infty} \int_E f_k\, dm = \int_E \lim_{k\to\infty} f_k\, dm = 0,$$

since $f_k \searrow 0$ as $k \to \infty$. This is true because $f(x) = 0$ for $x \in \bigcap\limits_{k=1}^{\infty} E_k$. $\square$

Problem 4: (30 points) Compute the following limit if it exists and justify the calculations. If the limit does not exist explain why it does not exist.

a) $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{\sqrt{n}}{\sqrt{x}}\cdot\chi_{[0,\frac{1}{n}]}\,dx$

SOLUTION. The function is nonnegative so the Lebesgue and Riemann integrals are the same, provided that the Riemann integral exists (as an improper integral in this case). We have

$$\int_0^1 \frac{\sqrt{n}}{\sqrt{x}}\cdot\chi_{[0,\frac{1}{n}]}\,dx = \int_0^{\frac{1}{n}} \frac{\sqrt{n}}{\sqrt{x}}\,dx = 2\sqrt{n}\sqrt{x}\Big|_0^{\frac{1}{n}} = 2\frac{\sqrt{n}}{\sqrt{n}} = 2.$$

Hence the limit is 2. □

b) $\displaystyle\lim_{n\to\infty}\int_a^\infty \frac{n\sin(\sqrt{x})}{1+n^2x^2}\,dx$, where $a>0$

SOLUTION. We have

$$\left|\frac{n\sin(\sqrt{x})}{1+n^2x^2}\right| \le \left|\frac{n}{1+n^2x^2}\right| < \left|\frac{n}{n^2x^2}\right| = \frac{1}{nx^2} \le \frac{1}{x^2},$$

since $n\ge 1$ and $1+n^2x^2 > n^2x^2$. The function $\frac{1}{x^2}$ is Lebesgue integrable on $[a,\infty)$ for $a>0$ since

$$\int_a^\infty \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_a^\infty = \frac{1}{a} < \infty.$$

As before, the function we are integrating is nonnegative so the Lebesgue and Riemann integrals are the same. Let $f_n(x) = \frac{n\sin(\sqrt{x})}{1+n^2x^2}$ for $n\ge 1$. We have shown that $|f_n(x)| < \frac{1}{x^2}$ and $\frac{1}{x^2}$ is Lebesgue integrable. Moreover, $f_n(x)\to 0$ as $n\to\infty$ pointwise for every x. By the Dominated Convergence Theorem it follows that

$$\lim_{n\to\infty}\int_a^\infty \frac{n\sin(\sqrt{x})}{1+n^2x^2}\,dx = 0.$$

□