Midterm 2 Practice Problems

Problem 1. Let the sequence (x_n) be defined as

$$x_n = \begin{cases} 1 + \frac{1}{n} & \text{if } n \text{ is odd;} \\ \frac{1}{n^2} & \text{if } n \text{ is even.} \end{cases}$$

Is (x_n) convergent?

Problem 2. Suppose $\lim_{n\to\infty} x_n = a > 0$. Prove that there exists a $K \in \mathbb{N}$ such that

$$\frac{a}{2} < x_n < 2a$$

for any $n \geq K$.

Problem 3.

1. Let the function f be defined as

   $$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   Prove that f is continuous at 0.

2. Let the function f be defined as

   $$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   Prove that f is discontinuous everywhere.

Problem 4. Give examples of functions f and g such that f and g do not have limits at c, but fg has the limit at c.

Problem 5. Suppose for any $x \in [-1, 1]$, $|f(x)| \leq 2|x|$. Prove that f is continuous at 0.

Problem 6. Let f be a continuous function on $[0, 1]$ such that $f(x) \in [0, 1]$ for any $x \in [0, 1]$. Prove that there exists a $c \in [0, 1]$ such that $f(c) = c$.

Problem 7.

1. Let (x_n) be a sequence such that $|x_{n+1} - x_n| < 2^{-n}$ for any $n \in \mathbb{N}$. Prove that (x_n) is convergent.

2. Is the result still true if we only assume $|x_{n+1} - x_n| < \frac{1}{n}$ for any $n \in \mathbb{N}$?

Problem 8. Let f and g be continuous functions on (a, b) such that $f(r) = g(r)$ for each rational number $r \in (a, b)$. Prove $f(x) = g(x)$ for all $x \in (a, b)$.

Problem 9.

1. Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some $k > 0$, then f is uniformly continuous on $[0, \infty)$.
2. Prove $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Problem 10. Let f be a continuous function on $[0, 1]$ such that $f(x) \in \mathbb{Q}$ for any $x \in [0, 1]$. Prove that f is constant.