MAT 319, Spring 2012
Solutions to Midterm 2

1. Prove that the function $f(x) = 4x - 5$ is continuous at every point.

   (a) using the sequential definition.
   Let $x_0 \in \mathbb{R}$ be arbitrary. To show that f is continuous at x_0, let (x_n) be a sequence in $\mathbb{R}$ converging to x_0. We must show that $(f(x_n))$ converges to $f(x_0)$. But since $\lim (x_n) = x_0$, our limit laws for sequences tell us that $\lim (4x_n) = 4x_0$, and thus $\lim (4x_n - 5) = 4x_0 - 5$. But this is exactly the statement that $\lim (f(x_n)) = f(x_0)$.

   (b) using the $\epsilon - \delta$ definition.
   Let $x_0 \in \mathbb{R}$ be arbitrary. Let $\epsilon > 0$, and set $\delta = \frac{\epsilon}{4}$. Suppose that $|x - x_0| < \delta$. Then

$$\begin{aligned} |f(x) - f(x_0)| &= |4x - 5 - 4x_0 + 5| \\ &= |4(x - x_0)| \\ &= 4|x - x_0| \\ &< 4\delta = \epsilon. \end{aligned}$$

   Thus, f is continuous at x_0.

2.

   (a) Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x \to +\infty} f(x) = 5$. Prove that f is bounded.
   First, since $\lim_{x \to +\infty} f(x) = 5$, we know that given any $\epsilon > 0$, there exists $\alpha > 0$ such that if $x > \alpha$, then $|f(x) - 5| < \epsilon$. In particular, take $\epsilon = 1$: there exists α such that $4 < f(x) < 6$ whenever $x > \alpha$.
   Now, since f is continuous on the closed, bounded interval, $[0, \alpha]$, it achieves a maximum value M and a minimum value m on this interval. $m \leq f(x) \leq M$ for all $x \in [0, \alpha]$.
   Set $\bar{m} = \min\{m, 4\}$ and $\bar{M} = \max\{M, 6\}$. Take $x \in [0, +\infty)$. If $x \leq \alpha$, then $\bar{m} \leq m \leq f(x) \leq M \leq \bar{M}$. If $x > \alpha$, then $\bar{m} \leq 4 < f(x) < 6 < \bar{M}$. Thus, in general, f is bounded below by $\bar{m}$ and bounded above by $\bar{M}$.

   (b) What happens if we drop the condition $\lim_{x \to +\infty} f(x) = 5$? Is it true that an arbitrary continuous function $f : [0, +\infty) \to \mathbb{R}$ is bounded? Explain your answer.
   If f is an arbitrary continuous function, there is no guarantee that it will be bounded on $[0, +\infty)$. As a counterexample, take the function $f(x) = x$, which is continuous, and obviously satisfies the limit

$$\lim_{x \to +\infty} f(x) = +\infty.$$

   Thus, this continuous function is necessarily unbounded.

3.

   (a) Let (x_n) be a bounded sequence. Show that $(\sin x_n)$ has a convergent subsequence.
   For all $x \in \mathbb{R}$, $|\sin x| \leq 1$. Therefore, $(\sin x_n)$ is a bounded sequence, bounded by 1. Thus, by Bolzano-Weierstrass, $(\sin x_n)$ has a convergent subsequence.

   (b) What if the sequence (x_n) is not bounded? Is it true that $(\sin x_n)$ has a convergent subsequence for an arbitrary sequence (x_n)? Explain your answer.
   Yes, one can still show the existence of a convergent subsequence. The proof given for part (a) still holds, as we never used the fact that (x_n) was bounded.

4.

(a) Prove that a function $f : \mathbb{R} \to \mathbb{R}$ can have at most 1 limit at $+\infty$.
The easiest way to prove this is using the sequential definition of limits. That way, it does not matter if the limits are finite or infinite. Suppose that f has two distinct limits at $+\infty$, denoted by A and B. Then, given any sequence (x_n) tending to $+\infty$, we must have $\lim (f(x_n)) = A$ and $\lim (f(x_n)) = B$. Since we know that limits of sequences are unique, it must be true that $A = B$.

(b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that has no (finite or infinite) limit at $+\infty$ whatsoever. Prove that the limit does not exist.
An example of such a function is $f(x) = \sin x$. To prove that this has no limit at $+\infty$, let $r_n = 2\pi n$ and $s_n = 2\pi n + \frac{\pi}{2}$. Both r_n and s_n are sequences that diverge to $+\infty$. However, $f(r_n) = \sin(2\pi n) = 0$ while $f(s_n) = \sin\left(2\pi n + \frac{\pi}{2}\right) = 1$. Since $f(r_n)$ and $f(s_n)$ converge to different limits, f has no limit as $x \to +\infty$.