MAT 319 - Foundations of Analysis

Final

Stony Brook University
Spring 2021
May 18, 2021

COMPLETION OF PROBLEM 1 IS MANDATORY

You do not have to print this exam. You can use your own paper or write on a tablet.

There are 7 problems on 7 pages (plus the cover sheets) in this exam.

Answer all questions below. You must show all of your work and provide **complete justifications** for all of your claims. Insufficient justifications will not receive full credit.

Leave all answers in exact form (that is, do not approximate π, square roots, and so on).

This is an **open book exam**. You may consult your textbook, lecture notes and homework, but no other source like cell phones, calculators or other electronic devices (unless directed to do so). Submission of solutions written by someone else, or found in some other textbook, or found on the internet is a violation to academic integrity. You **may not collaborate** with each other.

First, write your answers on paper or record your answers using electronic means (e.g., a tablet and a digital pen). If you write on paper, take clear photos/scans of your solutions. Submit your work to **Gradescope**. *Only in case there is an outage, you may email your work to* `dimitrios.ntalampekos@stonybrook.edu`.

You have 2 hours and 15 minutes to complete this exam. The exam starts at 11:15am.

You have 15 extra minutes to scan and upload the completed exam, so the exam is due at 1:45pm.

Penalties for delayed submissions:

1. Submissions between 1:45pm-2:00pm are subject to 10% penalty.
2. Submissions between 2:00-2:15pm are subject to 20% penalty.
3. Penalty increases by 10% every 15 minutes. Submissions past 2:45pm will not be accepted.

[2 pts] **Problem 1.** Completion of this problem is mandatory.

If you do not print the exam, then write and sign the following academic integrity statement ***before*** *proceeding with the exam. If you do not complete this problem* ***your exam will not be graded****.*

[1 pts] (i) **I certify that all the work I will do in this exam will be my own, and that I will not consult with other people or sources of any kind during this exam, other than the textbook and notes I have previously made.**

Name: ____

Student ID: ____

Signature: ____

[1 pts] (ii)

Place your student ID here and scan it

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[10 pts] **Problem 2.** (Include **all steps** in your solution. No credit for only stating the solution.)

[5 pts] (i) Find the limit of the sequence

$$\left(\frac{5n^2-2}{1+n^2}\right)$$

and then prove that the sequence converges to that number using only the definition of the limit.

Solution: We compute

$$\lim_{n\to\infty}\frac{5n^2-2}{1+n^2}=\lim_{n\to\infty}\frac{(5n^2-2)/n^2}{(1+n^2)/n^2}=\lim_{n\to\infty}\frac{5-2/n^2}{1/n^2+1}=\frac{5-0}{0+1}=5.$$

Now we give a proof using the definition of the limit. Let $\varepsilon>0$. We have

$$\left|\frac{5n^2-2}{1+n^2}-5\right|=\frac{7}{1+n^2}.$$

This is less than ε if $n>\sqrt{\left|\frac{7}{\varepsilon}-1\right|}$. We choose $N>\sqrt{\left|\frac{7}{\varepsilon}-1\right|}$. Then for $n\geq N$ we have

$$\left|\frac{5n^2-2}{1+n^2}-5\right|<\varepsilon.$$

This completes the proof.

[5 pts] (ii) Compute the following limit if it exists and otherwise explain why it does not exist. If you apply some theorem, then explain why it can be applied.

$$\lim_{x\to 0^+}(\ln(x+1))^x$$

Solution: We write

$$(\ln(x+1))^x=e^{x\ln(\ln(x+1))}.$$

By L'Hospital's rule

$$\lim_{x\to 0+}x\ln(\ln(x+1))\overset{0\cdot\infty}{=}\lim_{x\to 0+}\frac{\ln(\ln(x+1))}{1/x}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to 0+}\frac{\frac{1}{\ln(x+1)}\frac{1}{x+1}}{-\frac{1}{x^2}}=\lim_{x\to 0+}\frac{-x^2}{(x+1)\ln(x+1)}$$

$$\overset{\frac{0}{0}}{=}\lim_{x\to 0^+}\frac{-2x}{\ln(x+1)+1}=\frac{0}{0+1}=0.$$

By the continuity of the exponential function, we have

$$\lim_{x\to 0^+}\ln(x+1))^x=\lim_{x\to 0^+}e^{x\ln(\ln(x+1))}=e^0=1.$$

[10 pts] **Problem 3.** (Include **all steps** in your solution. No credit for only stating the solution.)

[5 pts] (i) Let $g\colon (2,3) \to \mathbb{R}$ be a function with $g(x) \neq 0$ for all $x \in (2,3)$. If

$$\lim_{x\to 2^+} g(x) = \infty$$

then using only the definition of the limit show that

$$\lim_{x\to 2^+} \frac{1}{g(x)} = 0.$$

Solution: Let $\varepsilon > 0$. The assumption $\lim_{x\to 2^+} g(x) = \infty$ is equivalent to saying that for each $M > 0$ there exists $\delta > 0$ such that $g(x) > M$ for $2 < x < 2+\delta$. We choose $M = 1/\varepsilon$, so there exists $\delta > 0$ such that $g(x) > 1/\varepsilon$ for $2 < x < 2+\delta$. In particular, this implies that $g(x) > 0$ for $2 < x < 2+\delta$. By taking reciprocals, we conclude that

$$\left|\frac{1}{g(x)} - 0\right| = \frac{1}{g(x)} < \varepsilon$$

for $2 < x < 2+\delta$. This implies that $\lim_{x\to 2^+} \frac{1}{g(x)} = 0$.

[5 pts] (ii) Let $f\colon (a,b) \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to b^-} f(x) = +\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = -\infty.$$

Prove that the range of f is all real numbers. That is, show that for every $k \in \mathbb{R}$ there exists $c \in (a,b)$ such that $f(c) = k$.

Hint: Use a theorem about continuous functions.

Solution: Let $k \in \mathbb{R}$. Since $\lim_{x\to a^+} f(x) = -\infty$, there exists $\delta_1 > 0$ such that $f(x) < k$ for $a < x < a+\delta_1$. Thus, there exists α with $a < \alpha < a+\delta_1$ such that $f(\alpha) < k$. Similarly, since $\lim_{x\to b^-} f(x) = +\infty$, there exists β with $f(\beta) > k$. The intermediate value theorem implies that there exists c between α and β such that $f(c) = k$.

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[10 pts] **Problem 4.** (Include **all steps** in your solution. No credit for only stating the solution.)

[5 pts] (i) Show that the function

$$g(x) = \frac{1}{5x+2}$$

is uniformly continuous on the interval $[0, \infty)$. In your proof you may use the definition of uniform continuity or some theorem proved in class that implies uniform continuity.

Solution: For $x, y \geq 0$ we have

$$|g(x) - g(y)| = \left| \frac{5y - 5x}{(5x+2)(5y+2)} \right| \leq \frac{5}{(5 \cdot 0 + 2)(5 \cdot 0 + 2)} |x - y| = \frac{5}{4}|x - y|.$$

This implies that g is a Lipschitz function so it is uniformly continuous.

[5 pts] (ii) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function with $f(0) = 1$ and

$$f'(x) = \frac{1}{1 + f(x)^2} + \frac{5}{2 + 3x^2}$$

for every $x \in \mathbb{R}$. Show that the inverse function f^{-1} exists (on an appropriate interval that you do not have to specify) and compute $(f^{-1})'(1)$.

Solution: Observe that $f'(x) > 0$ for each $x \in \mathbb{R}$. Thus, f is strictly increasing and one-to-one. Hence, $f\colon \mathbb{R} \to f(\mathbb{R})$ is a bijection and $f^{-1}\colon f(\mathbb{R}) \to \mathbb{R}$ exists. Since, $f(0) = 1$, we have

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{\frac{1}{1+1^2} + \frac{5}{2}} = \frac{1}{3}.$$

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[10 pts] **Problem 5.** (Include **all steps** in your solution. No credit for only stating the solution.)

[5 pts] (i) Consider the function

$$f(x) = |x^2 - 1|,$$

where $x \in \mathbb{R}$. Is this function continuous at $x = 1$? Moreover, is it differentiable at $x = 1$? Prove your claims either using some theorem or using the definitions.

Solution: The function f is continuous at 1 because it is the composition of the continuous functions $g(x) = |x|$ and $h(x) = x^2 - 1$. To check the differentiability at $x = 1$ we compute

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = 2$$

and

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{-(x^2 - 1)}{x - 1} = -2.$$

Therefore,

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

does not exist and f is not differentiable at 1.

[5 pts] (ii) Explain why the function $f(x) = |x^2 - 1|$, $x \in [-2, 3]$, has an absolute maximum and an absolute minimum and then find them.

Hint: What are the potential points of maximum or minimum?

Solution: The function f is continuous on the closed and bounded interval $[-2, 3]$, so it has an absolute maximum and an absolute minimum. The potential points of maximum and minimum are the points where f' does not exist, or the points where $f' = 0$, or the endpoints of the interval. We have

$$f'(x) = \begin{cases} 2x, & 1 < x \leq 3 \\ -2x, & -2 \leq x < 1 \\ \text{does not exist}, & x = 1 \end{cases}.$$

Note that $f'(x) = 0$ only when $x = 0$. Also, $f'(x)$ does not exist when $x = \pm 1$. We have $f(0) = 1$, $f(-1) = 0$, $f(1) = 0$, $f(-2) = 3$, $f(3) = 8$. Thus, the absolute maximum is 8 (corresponding to $x = 3$) and the absolute minimum is 0 (corresponding to $x = \pm 1$).

[10 pts] **Problem 6.** (Include **all steps** in your solution. No credit for only stating the solution.)

[5 pts] (i) Consider the function $f(x) = \ln x$, $x > 0$. For each $n \in \mathbb{N}$, find a formula for the n-th derivative $f^{(n)}(x)$ and prove your prediction using induction.

Solution: We have $f'(x) = 1/x$, $f''(x) = -1/x^2 = -x^{-3}$, $f'''(x) = 2x^{-2}, \ldots$. We predict that

$$f^{(n)}(x) = (-1)^{n-1}(n-1)! \cdot x^{-n}.$$

We prove this using induction. Note that the above formula is true for the base case $n = 1$. Suppose that the above formula is true for the index n. For the index $n+1$ we have

$$f^{(n+1)}(x) = (f^{(n)}(x))' = (-1)^{n-1}(n-1)! \cdot (-n)x^{-n-1} = (-1)^n n! \cdot x^{-n-1}.$$

Thus, the predicted formula is valid for the index $n+1$. This completes the induction.

[5 pts] (ii) Using Taylor's theorem for the function f with center at the point 1, approximate the number $\ln 2$ up to an error of 10^{-1}. Explain carefully. You can leave your answer as a sum of finitely many rational numbers.

Solution: By Taylor's theorem, for $x > 0$ and $n \geq 1$ we have

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1}$$

for some c between x and 1. Using the formulas of the derivatives, for $x = 2$ we have

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{(n-1)}\frac{1}{n} + \frac{(-1)^n c^{-n-1}}{n+1}$$

for some $c \in (1, 2)$. Thus, $\ln 2$ is approximately equal to

$$1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{(n-1)}\frac{1}{n}$$

up to an error of $\frac{(-1)^n c^{-n-1}}{n+1}$. Since $c \in (1, 2)$, we have

$$\left|\frac{(-1)^n c^{-n-1}}{n+1}\right| \leq \frac{1}{n+1}.$$

We are asked to allow for an error of 10^{-1}, so we require that $\frac{1}{n+1} \leq \frac{1}{10}$. This is true if $n \geq 9$. Thus,

$$\ln 2 \simeq 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9}$$

up to an error of 10^{-1}. (In fact, the above approximation gives $\sim 0.746...$ and the actual value is 0.693....)

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[10 pts] **Problem 7.** (Include **all steps** in your solution. No credit for only stating the solution.)

[5 pts] (i) Suppose that $f\colon [0,1] \to \mathbb{R}$ is a Riemann integrable function with $f(x) \geq 0$ for all $x \in [0,1]$, and f is not identically the zero function. Is it true that

$$\int_0^1 f(x)\,dx > 0?$$

Provide a brief explanation without proof.

Solution: This is not true. Consider the function f with $f(0) = 1$ and $f(x) = 0$ for $0 < x \leq 1$. This is not identically the zero function and $f(x) \geq 0$ for all $x \in [0,1]$. However,

$$\int_0^1 f(x)\,dx = 0.$$

(See lecture notes for a proof.)

[5 pts] (ii) Suppose that $f\colon [0,1] \to \mathbb{R}$ is a given Riemann integrable function with $\int_0^1 f(x)\,dx = 7$ and with $|f(x)| \leq 10$ for every $x \in [0,1]$. Using only the definition, prove that the function

$$g(x) = \begin{cases} f(x), & 0 \leq x < 1 \\ 319, & x = 1 \end{cases}$$

is Riemann integrable on $[0,1]$ and find the value of $\int_0^1 g(x)\,dx$.

Solution: We will show that g is Riemann integrable on $[0,1]$ and $\int_0^1 g(x)\,dx = 7$. Let $\varepsilon > 0$. Since f is Riemann integrable, there exists $\delta_1 > 0$ such that for each tagged partition $\dot{P}$ of $[0,1]$ with $\|\dot{P}\| < \delta_1$ we have $|S(f;\dot{P}) - 7| < \varepsilon/2$. Let $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ be a tagged partition of $[0,1]$. Then

$$\begin{aligned} S(g;\dot{P}) &= \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n-1} g(t_i)(x_i - x_{i-1}) + g(t_n)(x_n - x_{n-1}) \\ &= \sum_{i=1}^{n-1} f(t_i)(x_i - x_{i-1}) + g(t_n)(x_n - x_{n-1}) \\ &= S(f;\dot{P}) - f(t_n)(x_n - x_{n-1}) + g(t_n)(x_n - x_{n-1}) \\ &= S(f;\dot{P}) + (g(t_n) - f(t_n))(x_n - x_{n-1}). \end{aligned}$$

Thus, if $\|\dot{P}\| < \delta_1$, we have

$$|S(g;\dot{P}) - 7| \leq |S(f;\dot{P}) - 7| + |g(t_n) - f(t_n)|(x_n - x_{n-1}) < \varepsilon/2 + 329\|\dot{P}\|.$$

This is less than ε, if we require that $\|\dot{P}\| < \varepsilon/(2 \cdot 329)$. Thus, if we choose $\delta = \min\{\delta_1, \varepsilon/(2 \cdot 329)\}$, then, $|S(g;\dot{P}) - 7| < \varepsilon$ for $\|\dot{P}\| < \delta$. This completes the proof.