Problem 1. $y = c_1 x + c_2 x^3$ $\qquad y' = c_1 + 3c_2 x^2$

$$\begin{cases} c_1(-1) + c_2(-1)^3 = 1 \\ c_1 + 3c_2 = -2 \end{cases} \Longleftrightarrow \begin{cases} c_1 = -\frac{1}{2} \\ c_2 = -\frac{1}{2} \end{cases}$$

So. $y = -\frac{1}{2}(x + x^3)$

Problem 2. $y = c_1 x^2 + c_2 x^{-1}$ $\qquad y' = 2c_1 x - c_2 x^{-2}$

$$\begin{cases} c_1 + c_2 = 5 \\ 2c_1 - c_2 = -3 \end{cases} \Longleftrightarrow \begin{cases} c_1 = \frac{2}{3} \\ c_2 = \frac{13}{3} \end{cases}$$

$y = \frac{2}{3}x^2 + \frac{13}{3}x^{-1}$

Problem 3. $(2 + e^2 - 3\sin x) + (-2)(1 + 2e^2 - 3\sin x) + 3(e^2 - \sin x) = 0$

Problem 4. $$W = \begin{vmatrix} x^2 & \sin x & \cos x \\ 2x & \cos x & -\sin x \\ 2 & -\sin x & -\cos x \end{vmatrix} = x^2 \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} - \sin x \begin{vmatrix} 2x & -\sin x \\ 2 & -\cos x \end{vmatrix}$$

$$+ \cos x \begin{vmatrix} 2x & \cos x \\ 2 & -\sin x \end{vmatrix}$$

$$= -x^2 - \sin x(-2x\cos x + 2\sin x) + \cos x(-2x\sin x - 2\cos$$

$$= -x^2 - 2 \neq 0$$

Problem 5. 1. $r^3+2r^2+r=0$

$\Leftrightarrow r(r^2+2r+1)=0$

$\Leftrightarrow r(r+1)^2=0$

$y_1 = 1$

$y_2 = e^{-x}$

$y_3 = xe^{-x}$

$y = c_1 + c_2e^{-x} + c_3xe^{-x}$

$y' = -c_2e^{-x} + c_3e^{-x} - c_3xe^{-x}$

$y'' = c_2e^{-x} - c_3e^{-x} - c_3e^{-x} + c_3xe^{-x}$

$$\begin{cases} c_1 + c_2 = 2 \\ -c_2 + c_3 = -1 \\ c_2 - 2c_3 = 0. \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 2 \\ c_3 = 1 \end{cases}$$

so $y = 2e^{-x} + xe^{-x}$

2. $r^3 - 3r^2 + 3r - 1 = 0.$

$\Leftrightarrow (r-1)^3 = 0$

$$\begin{cases} y_1 = e^x \\ y_2 = xe^x \\ y_3 = x^2e^x \end{cases}$$

$y = (c_1 + c_2x + c_3x^2)e^x$

$y' = (c_2 + 2c_3x)e^x + (c_1 + c_2x + c_3x^2)e^x$

$= (c_1 + c_2 + (2c_3 + c_2)x + c_3x^2)e^x$

$y'' = (2c_3 + c_2 + 2c_3x)e^x + (c_1 + c_2 + (2c_3 + c_2)x + c_3x^2)e^x$

$= (c_1 + 2c_2 + 2c_3 + (4c_3 + c_2)x + c_3x^2)e^x$

$$\begin{cases} c_1 = 1 \\ c_1 + c_2 = 1 \\ c_1 + 2c_2 + 2c_3 = 2 \end{cases} \iff \begin{cases} c_1 = 1 \\ c_2 = 0 \\ c_3 = \frac{1}{2} \end{cases} \qquad y = \left(1 + \frac{x^2}{2}\right) e^x$$

Problem 6. 1. $r^3 - 8 = 0$

$$\iff (r-2)(r^2 + 2r + 4) = 0$$

$$\iff (r-2)\left((r+1)^2 + 3\right) = 0$$

$r_1 = 2$

$r_2 = -1 + \sqrt{3}\,i$

$r_3 = -1 - \sqrt{3}\,i$

$y_1 = e^{2x} \qquad y_2 = e^{-x}\cos\sqrt{3}x \qquad y_3 = e^{-x}\sin\sqrt{3}x$

$$y = c_1 e^{2x} + c_2 e^{-x}\cos\sqrt{3}x + c_3 e^{-x}\sin\sqrt{3}x.$$

2. $r^5 - r = 0 \iff r(r^4 - 1) = 0 \iff r(r-1)(r+1)(r^2+1) = 0$

$r_1 = 0 \quad r_2 = 1 \quad r_3 = -1 \quad r_4 = i \quad r_5 = -i$

$y_1 = 1 \quad y_2 = e^x \quad y_3 = e^{-x} \quad y_4 = \cos x \quad y_5 = \sin x$

$$y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x.$$

3. $r^3-5r^2+8r-4=0 \iff (r-1)(r-2)^2=0$

$y_1=e^x \qquad y_2=e^{2x} \qquad y_3=xe^{2x}$

$y=c_1e^x+c_2e^{2x}+c_3xe^{2x}$

4. $r^4+2r^2+1=0 \iff (r^2+1)^2=0$

$y_1=\cos x \qquad y_2=\sin x \qquad y_3=x\cos x \qquad y_4=x\sin x$

$y=c_1\cos x+c_2\sin x+c_3x\cos x+c_4x\sin x$

Problem 7. 1. $r^2-r-2=0 \iff (r-2)(r+1)=0$ $\qquad \begin{cases} y_1=e^{2x} \\ y_2=e^{-x} \end{cases}$

$y_c=c_1e^{2x}+c_2e^{-x}$

$W(y_1,y_2)=\begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x}, & -e^{-x} \end{vmatrix} = -3e^x$

$$y_p=-e^{2x}\int\frac{e^{-x}(3x+4)}{-3e^x}dx$$

$$+e^{-x}\int\frac{e^{2x}(3x+4)}{-3e^x}dx$$

$$=-\frac{3x}{2}-\frac{5}{4}$$

$y=y_p+y_c$

2. $r^2-4=0 \iff (r-2)(r+2)=0$

$y_1 = e^{2x} \quad y_2 = e^{-2x}$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$$

$$y_p = -e^{2x}\int \frac{e^{-2x}(2e^{2x})}{-4}dx + e^{-2x}\int \frac{e^{2x}(2e^{2x})}{-4}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{e^{2x}}{8}$$

$y_c = c_1 e^{-x}\cos x + C_2 e^{-x}\sin x \qquad y = y_p + y_c.$

3 $r^2+1=0 \quad y_1 = \cos x \quad y_2 = \sin x \quad W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$$y_p = -\cos x \int \frac{\sin x(\sin x + x\cos x)}{1}dx + \sin x \int \frac{\cos x(\sin x + x\cos x)}{1}dx$$

$$= -\cos x\left(\frac{x}{2} - \frac{\sin 2x}{8} - \frac{x\cos 2x}{4}\right) + \sin x\left(x\,\frac{\sin 2x}{4} - \frac{\cos 2x}{8} + \frac{x^2}{4}\right)$$

$y_c = C_1\cos x + C_2\sin x \qquad y = y_p + y_c$

Problem 8.

$r^2 - 7r + 6 = 0$ $\quad\quad y_1 = e^x \quad y_2 = e^{6x}$

$\Leftrightarrow (r-1)(r-6) = 0$

$$W = \begin{vmatrix} e^x & e^{6x} \\ e^x & 6e^{6x} \end{vmatrix} = 5e^{7x}$$

$$y_p = -e^x \int \frac{e^{6x}\sin 3x}{5e^{7x}}dx + e^{6x}\int \frac{e^x \sin(3x)}{5e^{7x}} dx$$

$$= -\frac{e^x}{5}\left(-\frac{3}{4}\right)\left(\frac{e^{-x}}{3}\sin 3x + e^{-x}\cos 3x\right)$$

$$+ \frac{e^{6x}}{5}\left(-\frac{2}{5}\right)\left(\frac{e^{-6x}}{3}\sin 3x + \frac{e^{-6x}}{6}\cos 3x\right) = \frac{17}{500}\sin 3x + \frac{41}{300}\cos 3x$$

$$y_c = c_1 e^x + c_2 e^{6x}$$

$$y = c_1 e^x + c_2 e^{6x} + \frac{17}{500}\sin 3x + \frac{41}{300}\cos 3x$$

$$y' = c_1 e^x + 6c_2 e^{6x} + \frac{51}{500}\cos 3x - \frac{41}{100}\sin 3x$$

$$\begin{cases} c_1 + c_2 + \frac{41}{300} = 3 \\ c_1 + 6c_2 + \frac{51}{500} = 2 \end{cases} \quad \begin{cases} c_1 = \frac{362}{1875} + 3 - \frac{41}{300} \\ c_2 = -\frac{362}{1875} \end{cases}$$