Practice Midterm I

Solutions

MAT 303 : Calculus IV

Problem 1 (i).

$$\frac{dy}{dx} = xy - \frac{x}{y} \qquad \text{Separable}$$

$$\frac{dy}{dx} = \frac{xy^2 - x}{y} = \frac{x(y^2-1)}{y}$$

$$\frac{y}{y^2-1}\,dy = x\,dx$$

$$\int \frac{y}{y^2-1}\,dy = \int x\,dx + C = \frac{x^2}{2} + C$$

$\int$ substitution

$u = y^2 - 1$

$du = 2y\,dy$

$$\int \frac{du/2}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|y^2-1|$$

Hence $\frac{1}{2}\ln|y^2-1| = \frac{x^2}{2} + C$

$$\ln|y^2-1| = x^2 + 2C$$

$$|y^2-1| = e^{x^2+2C} = e^{x^2}e^{2C}$$

$$Y^2 - 1 = \pm e^{x^2} e^{2C}$$

Set $c' = \pm e^{2C}$ constant

$$Y^2 - 1 = c' e^{x^2}$$

$$Y^2 = c' e^{x^2} + 1$$

$$Y = \pm \sqrt{c' e^{x^2} + 1}$$

Problem 1 (ii)

$$\frac{dY}{dx} = \sqrt{x + Y + 1}$$

Use a substitution.

$$u^2 = x + Y + 1$$

$$Y = u^2 - x - 1$$

$$\frac{dY}{dx} = 2u \frac{du}{dx} - 1$$

Rewrite the equation:

$$2u \frac{du}{dx} - 1 = \sqrt{u^2} = u$$

$$2u \frac{du}{dx} = u + 1 \qquad \text{Separable}$$

$$\frac{2u}{u+1}\, du = dx$$

$$\int \frac{2u}{u+1}\,du = \int dx + C = x + C$$

$$=$$

$$2\int \frac{u}{u+1}\,du$$

$$=$$

$$2\int \frac{u+1-1}{u+1}\,du = 2\int du - 2\int \frac{1}{u+1}\,du =$$

$$= 2u - 2\ln|u+1|$$

Hence

$$2u - 2\ln|u+1| = x + C$$

But $u = \sqrt{x+y+1}$ and so

$$2\sqrt{x+y+1} - 2\ln\left|\sqrt{x+y+1}+1\right| = x + C$$

Leave this solution in implicit form.

Problem 1 (iii)

$$(x+2y)\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = \frac{y}{x+2y} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{y/x}{1+2\left(\frac{y}{x}\right)}$$

Homogenous equation

substitution: $v = \frac{y}{x}$

$$y = vx \quad \text{and} \quad \frac{dy}{dx} = \frac{dv}{dx}x + v$$

Rewrite the equation:

$$\frac{dv}{dx}x + v = \frac{v}{1+2v}$$

$$\frac{dv}{dx}x = \frac{v}{1+2v} - v = \frac{v - v - 2v^2}{1+2v}$$

Separable

$$\frac{1+2v}{v^2}\,dv = \frac{1}{x}\,dx$$

$$\int \frac{1+2v}{v^2}\,dv = \int \frac{1}{x}\,dx + C = \ln|x| + C$$

$$\shortparallel$$

$$\int \frac{1}{v^2}\,dv + 2\int \frac{1}{v}\,dv$$

$$\shortparallel$$

$$-\frac{1}{v} + 2\ln|v|$$

Hence

$$-\frac{1}{v} + 2\ln|v| = \ln|x| + C$$

But $v = \frac{y}{x}$ and so

$$-\frac{x}{y} + 2\ln\left|\frac{y}{x}\right| = \ln|x| + C$$

We can leave this solution in implicit form.

Problem 1 (iv)

$$\frac{dy}{dx} = x - \frac{1}{x^2 - 2y}$$

Substitution $u = x^2 - 2y$

$$y = \frac{1}{2}x^2 - \frac{1}{2}u$$

$$\frac{dy}{dx} = x - \frac{1}{2}\frac{du}{dx}$$

Rewrite the equation

$$\overbrace{\cancel{x} - \frac{1}{2}\frac{du}{dx}}^{dy/dx} = \cancel{x} - \frac{1}{u}$$

$$\frac{1}{2}\frac{du}{dx} = \frac{1}{u} \quad \text{Separable}$$

$$u\,du = 2\,dx$$

$$\int u\,du = \int 2\,dx + C$$

$$\frac{u^2}{2} = 2x + C$$

$$u^2 = 4x + 2C$$

$$u = \pm\sqrt{4x + 2C}$$

But $v = x^2 - 2y$, and so

$$x^2 - 2y = \pm\sqrt{4x + 2C}$$

$$-2y = -x^2 \pm \sqrt{4x + 2C}$$

$$y = \frac{x^2}{2} \pm \frac{1}{2}\sqrt{4x + 2C}$$

Problem 3 (i)

$$2x\frac{dy}{dx} - 3y = 9x^3$$

$$\frac{dy}{dx} - \frac{3}{2x}y = \frac{9}{2}x^2$$

Linear

$P(x) = -\frac{3}{2x}$

$Q(x) = \frac{9}{2}x^2$

$$\int P(x)\,dx = \int -\frac{3}{2x}\,dx$$

$$= -\frac{3}{2}\ln|x|$$

The integrating factor is

$$e^{\int P(x)\,dx} = e^{-3/2\ln|x|} = |x|^{-3/2}$$

Now ~~the~~ because there is an absolute value, we have to distinguish 2 cases: $x \geq 0$ and $x < 0$.

We only work out the ~~one~~ case $x \geq 0$.

Hence $|x| = x$ and

$$e^{\int p(x)\,dx} = x^{-3/2}$$

Multiply by $x^{-3/2}$:

$$x^{-3/2} \frac{dy}{dx} - x^{-3/2} \frac{3}{2x} y = x^{-3/2} \frac{9}{2} x^2$$

$$\frac{d}{dx}\left(x^{-3/2} y\right) = \frac{9}{2} x^{-3/2+2} = \frac{9}{2} x^{1/2}$$

Integrating

$$x^{-3/2} y = \int \frac{9}{2} x^{1/2} + C$$

$$= \frac{9}{2} \frac{x^{3/2}}{3/2} + C$$

$$= 3 x^{3/2} + C$$

$$\boxed{\begin{aligned} y &= 3 x^{3/2} \cdot x^{3/2} + C x^{3/2} = \\ &= 3x^3 + C x^{3/2} \end{aligned}}$$

Problem 3 (iii)

$$x\frac{dy}{dx} + y = 3xy \qquad y(1) = 0.$$

This is Linear.

First we rewrite it in standard form $\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$

~~Note~~ The RHS is a term with the y, so we move it to the LHS.

$$x\frac{dy}{dx} + y - 3xy = 0$$

$$x\frac{dy}{dx} + (1-3x)y = 0$$

$$\frac{dy}{dx} + \frac{1-3x}{x}y = 0 \quad \leftarrow \text{Standard form}$$

$$P(x) = \frac{1-3x}{x}$$

$$Q(x) = 0$$

$$\int P(x)\,dx = \int \frac{1-3x}{x}\,dx = \int \frac{1}{x}\,dx - 3\int dx$$

$$= \ln|x| - 3x$$

The integrating factor is

$$e^{\int P(x)\,dx} = e^{\ln|x| - 3x} = |x|\, e^{-3x}$$

Since the initial data $Y(1) = 0$ has $x_0 = 1 > 0$, we can assume that x is positive because we are looking for a solution near $x_0 = 1$.

Hence $|x| = x$ and

$$e^{\int P(x)dx} = x e^{-3x}$$

$$x e^{-3x} \frac{dY}{dx} + x e^{-3x} \frac{1-3x}{x} Y = 0$$

$$\frac{d}{dx}\left(x e^{-3x} Y\right) = 0$$

Integrating:

$$x e^{-3x} Y = \int 0\, dx + C = C$$

$$Y = \frac{C}{x e^{-3x}}$$

Impose the initial condition:

$x = 1 \quad , \quad y = 0$

$$0 = \frac{c}{1} \longrightarrow c = 0$$

The particular solution is

$$y = 0$$

Problem 3 (iii)

$$x^3 \frac{dy}{dx} = xy + 1 \quad , \qquad y(1) = 0$$

Write this in the standard form

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

$$x^3 \frac{dy}{dx} - xy = 1$$

$$\frac{dy}{dx} - \frac{1}{x^2}\, y = \frac{1}{x^3}$$

Linear with

$P(x) = -\frac{1}{x^2}$

$Q(x) = \frac{1}{x^3}$

$$\int P(x)\, dx = \int -\frac{1}{x^2}\, dx =$$

$$= \frac{1}{x}$$

The integrating factor is

$$e^{\int P(x)\,dx} = e^{1/x}$$

$$e^{1/x}\frac{dY}{dx} - e^{1/x}\frac{1}{x^2}Y = e^{1/x}\frac{1}{x^3}$$

$$\frac{d}{dx}\left(e^{1/x}Y\right) = e^{1/x}\frac{1}{x^3} \quad \leftarrow \text{Integrating}$$

$$e^{1/x}Y = \int \frac{1}{x^3}e^{1/x}dx + C$$

For the integral use a substitution:

$$u = \frac{1}{x}$$

$$du = -\frac{1}{x^2}\,du$$

Then $\int e^{1/x}\frac{1}{x^3}\,dx = \int e^{1/x}\left(-\frac{1}{x^2}\right)\left(-\frac{1}{x}\right)dx$

$$= \int e^u(-u)\,du = -\int u e^u\,du$$

By parts

$$= -ue^u + \int e^u = -ue^u + e^u$$

$$= -\frac{1}{x}e^{1/x} + e^{1/x}$$

$$\cancel{\frac{d}{dx}\left(e^{1/x} y\right) = -\frac{1}{x} e^{1/x} + e^{1/x} + A}$$

~~Integrating~~

Hence:

$$e^{1/x} y = -\frac{1}{x} e^{1/x} + e^{1/x} + C$$

Multiply by $e^{-1/x}$

$$y = -\frac{1}{x} e^{1/x} e^{-1/x} + e^{1/x} e^{-1/x} + C e^{-1/x}$$

$$y = -\frac{1}{x} + 1 + C e^{-1/x}$$

Impose initial data: $x = 1$, $y = 0$

$$0 = -1 + 1 + C$$

$$C = 0$$

The particular solution is

$$y = -\frac{1}{x} + 1$$

Problem 4 (ii)

$$\underbrace{\left(x + \arctan y\right)}_{\substack{\shortparallel \\ M}} dx + \underbrace{\left(\frac{x+y}{1+y^2}\right)}_{\substack{\shortparallel \\ N}} dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{1+y^2}\right) + \frac{\partial}{\partial x}\left(\frac{y}{1+y^2}\right) = \cancel{0}$$

$$= \frac{1}{1+y^2} + 0 \qquad 6$$

Hence $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and the equation is exact.

Then there exists $F(x,y)$ such that $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$

Integrating wrt x:

$$F(x,y) = \int M\,dx + g(y) \qquad 6$$

$\hookrightarrow$ function of y

$$= \int (x + \arctan y)\, dx + g(y)$$

$$= \frac{x^2}{2} + x \arctan y + g(y)$$

$$\frac{\partial F}{\partial y} = \frac{x}{1+y^2} + g'(y) \overset{!}{=} N = \frac{x+y}{1+y^2}$$

Hence $g'(y) = \dfrac{x+y}{1+y^2} - \dfrac{x}{1+y^2} =$

$$= \frac{x + y - x}{1+y^2}$$

$$= \frac{y}{1+y^2}$$

$$g(y) = \int \frac{y}{1+2y^2}\, dy + C =$$

Use a subst. $u = 1 + 2y^2$

$du = 4y\, dy$

$$= \int \frac{du/4}{u} = \frac{1}{4} \int \frac{du}{u} = \frac{1}{4} \ln|u| + C =$$

$$= \frac{1}{4} \ln|y^2 + 1| + C$$

Then the solution is

$$F(x,y) = \frac{x^2}{2} + x \arctan y +$$
$$+ \frac{1}{4} \ln|y^2+1| + C = 0.$$

Problem ~~4~~ (iii)

$$\underbrace{\left(e^x \sin y + \tan y\right)}_{\overset{\shortparallel}{M}} dx + \underbrace{\left(e^x \cos y + x \sec^2 y\right)}_{\overset{\shortparallel}{N}} dy = 0$$

$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y$$

$$\frac{\partial N}{\partial x} = e^x \cos y + \sec^2 y$$

$=$ proves exactness

Therefore there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

$$F(x,y) = \int M\,dx + g(y)$$

$$= \int (e^x \sin y + \tan y)\,dx + g(y)$$

$$= e^x \sin y + x \tan y + g(y)$$

$$\frac{\partial F}{\partial y} = e^x \cos y + x \sec^2 y + g'(y)$$

$$\stackrel{!}{=} N = e^x \cos y + x \sec^2 y$$

Hence $g'(y) = 0$ and $g(y) = \int 0\,dx + C$

$$= C$$

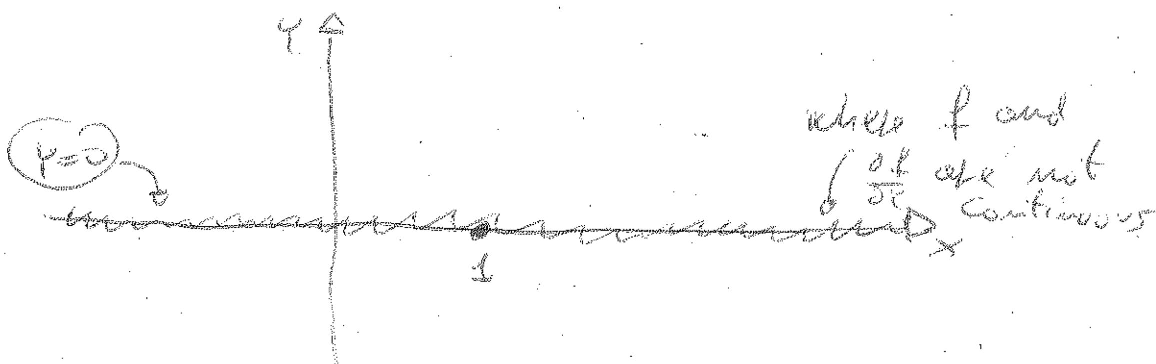
$$F(x,y) = e^x \sin y + x \tan y + C = 0$$

Problem 5 (i)

$$\frac{dy}{dx} = \frac{x-1}{y} \qquad \underset{x_0}{y(1)} = \underset{y_0}{0}$$

$f(x,y) = \dfrac{x-1}{y} \longrightarrow$ not continuous on $y=0$

$$\frac{\partial f}{\partial y} = (x-1)\left(-\frac{1}{y^2}\right)$$



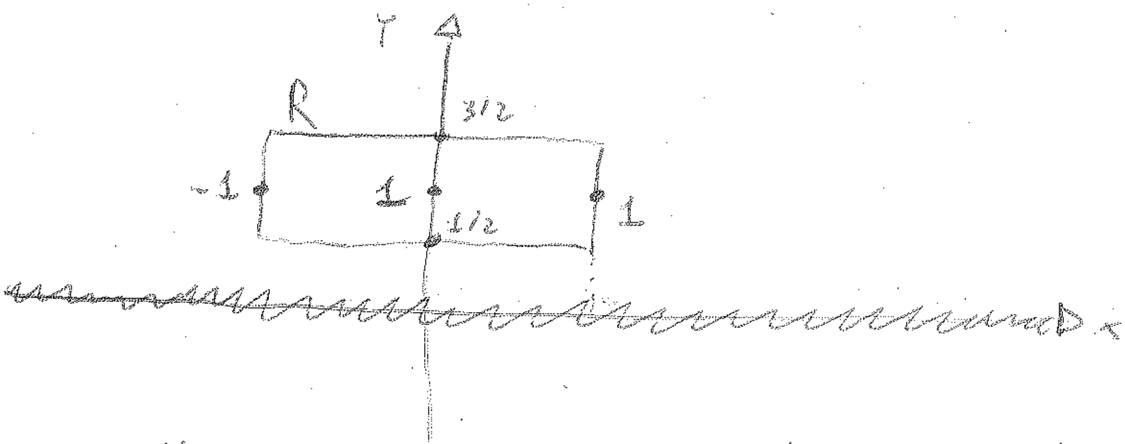
I can't find a rectangle containing $(1,0)$ and where both f and $\dfrac{\partial f}{\partial y}$ are continuous.

Problem 5 (ii)

$$\frac{dy}{dx} = \frac{x-1}{y} \qquad y(\underset{x_0}{\overset{\parallel}{0}}) = \underset{y_0}{\overset{\parallel}{1}}$$

$$f(x,y) = \frac{x-1}{y} \longrightarrow \text{not continuous on } y=0$$

$$\frac{\partial f}{\partial y} = (x-1)\left(-\frac{1}{y^2}\right)$$

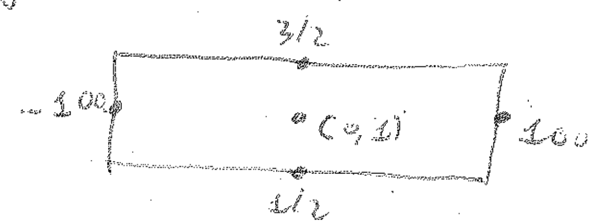


On the rectangle R both f and $\frac{\partial f}{\partial y}$ are continuous.

Hence there is a unique solution.

In fact any ~~the~~ symmetric rectangle containing $(0,1)$ and avoiding the x-axis works.

For instance



Problem 6

a = ~~k~~ acceleration

$\frac{dv}{dt} = a$ gives

$$v(t) = at + v_0 = at + 88$$

$\frac{dv}{dt} = s$ (= position function) gives

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0$$

Can assume $s_0 = 0$ because we start counting time since the moment the brakes are applied.

The time at which the car stops is given by solving $v(t) = 0$

Hence $at + 88 = 0$

and $t = -\frac{88}{a}$

The car skids 176 ft it means that

$$s\left(-\frac{88}{a}\right) = 176$$

Hence $\frac{1}{2}a\frac{(-88)^2}{a^2} + 88\left(-\frac{88}{a}\right) = 176$

$$\frac{1}{2}\frac{88^2}{a} - \frac{88^2}{a} = 176$$

$$-\frac{1}{2}\frac{88^2}{a} = 176$$

$$a = -\frac{88^2}{2\cdot(176)} = -22 \text{ ft}/s^2$$

Problem 7

$$\begin{cases} \dfrac{dA}{dt} = kA \\ A(0) = 10 \end{cases}$$

~~A(t) triple~~ $\cancel{A(7.5) = 3\cdot A(0) = 3(10)}$

This is the exponential model.

The solution is

$$A(t) = 10e^{kt}$$

$$A(7.5) = 3\cdot A(0) = 3(10) = 30$$

A(t) is tripling every 7.5 years

$$10e^{k(7.5)} = 30$$

$$e^{7.5k} = 3$$

$$(7.5)k = \ln 3$$

$$k = \frac{1}{7.5}\ln(3)$$

So $$A(t) = 10 \cdot \left(3^{(1/7.5)t}\right)$$

How long does it take for $A(t)$ to centuplicate?

Need to solve $A(t) = 100 \cdot A(0)$

$$= 100(10)$$

$$= 1000$$

$$10 \cdot 3^{\left(\frac{1}{7.5}t\right)} = 1000$$

$$3^{\left(\frac{1}{7.5}t\right)} = 100$$

$$\frac{1}{7.5}t = \ln_3 100$$

$$t = (7.5)\left(\ln_3(100)\right)$$

Problem 9

$$\begin{cases} \frac{dT}{dt} = k(A - T) \\ \cancel{\frac{dT}{dt}}\ T(0) = 210 \end{cases}$$

$A = 70$

$$\begin{cases} \frac{dT}{dt} = k(70 - T) \\ T(0) = 210 \end{cases}$$

Solve this IVP

Separable

$$\frac{dT}{70-T} = k\,dt$$

$$\int \frac{dT}{70-T} = \int k\,dt + C$$

$$-\ln(T-70) = kt + C$$

$$\ln(T-70) = -kt - C$$

$$T - 70 = e^{-kt - c} = e^{-kt} e^{-c}$$

Let $c' = e^{-c}$

$$T = 70 + c' e^{-kt}$$

Initial condition $t = 0$ $T = 210$

$$210 = 70 + c' \rightsquigarrow c' = 140$$

So $T = 70 + 140 e^{-kt}$

(After 30 min the cake is at 140°F.)

To find k need to solve

$$140 = T(30) = 70 + 140 e^{(-30k)}$$

$$70 = 140 e^{-30k}$$

$$e^{-30k} = \frac{1}{2}$$

$$-30k = \ln\left(\frac{1}{2}\right)$$

$$k = -\frac{1}{30} \ln\left(\frac{1}{2}\right)$$

When will the cake be 100°F?

Need to solve $T(t) = 100$

$$\cancel{T(t) = 70 + 140\left(\tfrac{1}{2}\right)^{t/30} = 100}$$

$$\cancel{140\left(\tfrac{1}{2}\right)^{t/30} = 100}$$

$$\cancel{\left(\tfrac{1}{2}\right)^{t/30} = \frac{100}{140} = \frac{10}{14} = \frac{5}{7}}$$

$$\cancel{\frac{t}{30} = \ln_2\left(\tfrac{5}{7}\right)}$$

$$\cancel{t = 30 \ln_2\left(\tfrac{5}{7}\right)}$$

$$\cancel{T(t) = 70 + 140\left(\tfrac{1}{2}\right)}$$

$$T(t) = 70 + 140\left(\frac{1}{2}\right)^{t/30} = 100$$

$$140\left(\frac{1}{2}\right)^{t/30} = 30$$

$$\left(\frac{1}{2}\right)^{t/30} = \frac{30}{140} = \frac{3}{14}$$

$$t/30 = \ln_{1/2}\left(\frac{3}{14}\right)$$

$$t = 30 \ln_{\frac{1}{2}}\left(\frac{3}{14}\right).$$

Problem 8

$V_0 = 60$

$c_{input} = 1$ [illegible]

$r_{input} = 2$ gal/min

$r_{out} = 3$ gal/min

After 60 minutes the tank is empty.

t = time in minutes

$x(t)$ = quantity of salt in the tank.

$$c_{out} = \frac{x}{V}$$

$$V = \text{volume} = V_0 + (r_{in} - r_{out})t$$
$$= 60 - t.$$

$$c_{out} = \frac{x}{60 - t}$$

The equation to solve is:

$$\frac{dx}{dt} = c_{in} r_{in} - c_{out} r_{out} = 2 - \frac{3x}{60 - t}$$

$$\frac{dx}{dt} + \frac{3x}{60-t} = 2$$

Linear with $P(t) = \dfrac{3}{60-t}$

$Q(t) = 2$

$$\int P(t)\,dt = 3\int \frac{dt}{60-t} = -3\ln(t-60)$$

$$e^{\int P(t)\,dt} = e^{-3\ln(t-60)} = (t-60)^{-3}$$

$$(t-60)^{-3}\frac{dx}{dx} + (t-60)^{-3}\frac{3x}{60-t} = (t-60)^{-3}\cdot 2$$

$$\frac{d}{dt}\left((t-60)^{-3}x\right) = \frac{2}{(t-60)^3}$$ Integrating

$$(t-60)^{-3}x = \int \frac{2}{(t-60)^3}\,dt = 2\,\frac{(-2)}{(t-60)^2} + C$$

$$x = -4(t-60) + C(t-60)^3$$

Initial condition: $x(0) = 0$

$$0 = -4(-60) + C(-60)^3$$

$$C = \frac{-60\cdot 4}{(-60)^3} = \frac{4}{3600} = \frac{1}{900}$$

Hence

$$x(t) = -4(t-60) + \frac{1}{900}(t-60)^3$$

To find at what time there was the maximum quantity of salt, we need to find the max of the function $x(t)$.

$$\frac{dx}{dt} = -4 + \frac{3}{900}(t-60)^2 = 0$$

$$\frac{1}{300}(t-60)^2 = 4$$

$$(t-60)^2 = 4 \cdot 300 = 1200$$

$$t - 60 = \pm\sqrt{1200}$$

$$t = \pm\sqrt{1200} + 60$$

The solution with + is to be excluded.

Hence $t = 60 - \sqrt{1200}$ min.