Problem 1

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1. $\frac{y}{1+y^2}y' = x \iff \int \frac{y}{1+y^2}\,dy = \int x\,dx$

$\iff \frac{1}{2}\ln(1+y^2) = \frac{x^2}{2} + c$

$\iff 1+y^2 = e^{2\left(\frac{x^2}{2}+c\right)} = e^{x^2+2c}.$

2. $y' + \frac{1}{1+x}y = \frac{\cos x}{1+x}.$

$m = e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1+x$

$y = \frac{1}{m}\left(\int m\frac{\cos x}{1+x}dx\right) = \frac{1}{1+x}\int \cos x\,dx = \frac{1}{1+x}(\sin x + c)$

3. $y' = -\frac{y(3x+y)}{x(x+y)}$ set $v = \frac{y}{x}.$

$y' = v + xv' = -v\cdot\frac{3+v}{1+v} \iff xv' = -\frac{2v^2+4v}{1+v}.$

$\int \frac{1+v}{2v^2+4v}dv = \int \frac{1}{x}dx = \ln|x| + c.$

On the other hand, $\int \frac{1+v}{2v^2+4v}dv = \frac{1}{4}\int \frac{1}{v} + \frac{1}{v+2}dv = \frac{1}{4}\left(\ln|v| + \ln|v+2|\right)$

Therefore. $\ln|v^2+2v| = \ln|x|^4 + 4c \iff v^2+2v = c_1 x^4$

$(v+1)^2 = 1 + C_1 x^4$ $\qquad$ $y = vx = x\left(-1 \pm \sqrt{1+C_1 x^4}\right)$

4. $v = x + y + 2$

$v' = y' + 1 = \sqrt{v} + 1 \iff \int \frac{1}{1+\sqrt{v}}\,dv = \int 1\,dx = x + C$

$\int \frac{1}{1+\sqrt{v}}\,dv \overset{z=\sqrt{v}}{=\!=} \int \frac{2z\,dz}{1+z} = \int 2 - \frac{2}{1+z}\,dz = 2z - 2\ln|1+z|$

$= 2\sqrt{v} - 2\ln(1+\sqrt{v})$

Therefore. $\qquad 2\left(\sqrt{x+y+2} - \ln\left(1+\sqrt{x+y+2}\right)\right) = x + C$.

5. $y' + \frac{6}{x}y = 3y^{\frac{4}{3}}$ $\qquad$ $v = y^{1-\frac{4}{3}} = y^{-\frac{1}{3}}$

$v' = -\frac{1}{3}y^{-\frac{4}{3}}y' = -\frac{1}{3}y^{-\frac{4}{3}}\left(3y^{\frac{4}{3}} - \frac{6}{x}y\right) = -1 + \frac{2}{x}y^{-\frac{1}{3}} = \frac{2}{x}v - 1$

It is easy to see $\quad v = x^2 \int -x^{-2}\,dx = x^2(x^{-1} + C) = x + Cx^2$

So $\quad y = v^{-3} = (x + Cx^2)^{-3}$

6. $M = \cos x + \ln y \qquad N = \frac{x}{y} + e^y$

$M_y = N_x = \frac{1}{y}$ Therefore the ODE is exact.

Then we solve
$$\begin{cases} F_x = \cos x + \ln y & \text{--- ①} \\ F_y = \frac{x}{y} + e^y & \text{--- ②} \end{cases}$$

From ①. $F = \sin x + x \ln y + g(y)$

From ② $F_y = \frac{x}{y} + g'(y) = \frac{x}{y} + e^y$ so. $g = e^y$.

Therefore: the solution is $\sin x + x \cdot \ln y + e^y = C$.

Problem 2. we ~~denote~~ set $m(t)$ to be the amount of salt at t

and $V(t) = 100 + 2t$ to be the volumn of the brine at t.

$$\begin{cases} m(0) = 50 \\ m'(t) = 1 \times 5 - \frac{m(t)}{V(t)} \times 3 = 5 - \frac{3m(t)}{100+2t} \end{cases}$$

It is easy to see $m(t) = (100+2t)^{-\frac{3}{2}} \int 5(100+2t)^{\frac{3}{2}} dt$

$= (100+2t) + C(100+2t)^{-\frac{3}{2}}$

Since $m(0) = 50$, $C = -50000$ and

$$m(t) = 100 + 2t - 50000(100+2t)^{-\frac{3}{2}}$$

If t_0 is the time when the tank is full, then.

$100 + 2t_0 = 400$ so $t_0 = 150$.

therefore $m(t_0) = 100 + 150\times 2 - 50000(100 + 150\times 2)^{-\frac{3}{2}}$

$$= 400 - \frac{25}{4} \text{ (lb)}.$$

Problem 3.

1. $r^2 - 3r + 2 = 0 \Leftrightarrow (r-1)(r-2) = 0$ ~~(2r+1)(r~~

$y = c_1 e^x + c_2 e^{2x}$

2. $4r^2 + 4r + 1 = 0 \Leftrightarrow (2r+1)^2 = 0$

$y = (c_1 + c_2 x) e^{-\frac{x}{2}}$

3. $r^2 + 6r + 10 = 0 \Leftrightarrow (r+3)^2 = -1$ $r = -3 \pm i$

$y = c_1 e^{-3x} \cos x + c_2 e^{-3x} \sin x$

5. $r^3+3r^2+3r+1=0 \iff (r+1)^3=0$

$$y=(c_1+c_2x+c_3x^2)e^{-x}$$

Problem 4. 1. $r^3=1 \iff (r-1)(r^2+r+1)=0$

$$\iff (r-1)\left((r+\tfrac{1}{2})^2+\tfrac{3}{4}\right)=0$$

$$r_1=1 \qquad \begin{matrix} r_2 \\ r_3 \end{matrix} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$y=c_1e^x+c_2e^{-\frac{x}{2}}\cos\frac{\sqrt{3}}{2}x+c_3e^{-\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$$

since $y(0)=1$, $y'(0)=y''(0)=0$

$$\begin{cases} c_1+c_2=1 \\ c_1-\frac{c_2}{2}+\frac{\sqrt{3}}{2}c_3=0 \\ c_1-\frac{1}{2}\left(\frac{\sqrt{3}}{2}c_3-\frac{c_2}{2}\right)+\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}c_2-\frac{c_3}{2}\right)=0 \end{cases} \Longrightarrow \begin{cases} c_1=\frac{1}{3} \\ c_2=\frac{2}{3} \\ c_3=0 \end{cases}$$

2. $r^2+2r+2=0 \iff (r+1)^2=-1 \qquad r=-1\pm i$

$$y_c=c_1e^{-x}\cos x+c_2e^{-x}\sin x \qquad W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=e^{-2x}$$

$$=c_1y_1+c_2y_2$$

$$y_p = -e^{-x}\cos x \int \frac{e^{-x}\sin x \cancel{\sin x}\, e^{-x}}{e^{-2x}}\,dx + e^{-x}\sin x \int \frac{e^{-x}\cos x \cdot e^{-x}}{\cancel{e^{-2x}}}\,dx$$

$$= -e^{-x}\cos x(-\cos x) + e^{-x}\sin^2 x = e^{-x}$$

$$y = e^{-x} + c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$$

$$\begin{cases} y(0)=1 \\ y'(0)=2 \end{cases} \Longrightarrow \begin{cases} 1+c_1=1 \\ -1-c_1+c_2=2 \end{cases} \Longrightarrow \begin{cases} c_1=0 \\ c_2=3. \end{cases}$$

problem 5. Set $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B=\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$

$$\det(AB) = \det\begin{pmatrix} aa_1+bc_1 & ab_1+bd_1 \\ ca_1+dc_1 & cb_1+dd_1 \end{pmatrix} = (aa_1+bc_1)(cb_1+dd_1) - (ca_1+dc_1)(ab_1+bd_1)$$

$$= aca_1b_1 + ada_1d_1 + bcb_1c_1 + bdc_1d_1 - (aca_1b_1 + bca_1d_1 + adc_1b_1 + bdc_1d_1)$$

$$= (ad-bc)(a_1d_1-b_1c_1) = \det A \cdot \det B$$

Problem 6. proof: Set $A=(a_{ij})_{1\le i,j\le n}$ and $B=(b_{ij})_{1\le i,j\le n}$.

$$AB=\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)_{1\le i,j\le n}.$$

$$(AB)^T_{ij} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = B^TA^T$$

Problem 7. 1.

$$\begin{pmatrix} 2 & 3 & 2 & 3 \\ 4 & -5 & 5 & -7 \\ -3 & 7 & -2 & 5 \end{pmatrix} \xrightarrow[③+\frac{3}{2}①]{②-①} \begin{pmatrix} 2 & 3 & 2 & 3 \\ 0 & -11 & 1 & -13 \\ 0 & \frac{23}{2} & 1 & \frac{13}{2} \end{pmatrix}$$

$$\xrightarrow{③\times 2} \begin{pmatrix} 2 & 3 & 2 & 3 \\ 0 & -11 & 1 & -13 \\ 0 & 23 & 1 & 13 \end{pmatrix} \xrightarrow{③+\frac{23}{11}②} \begin{pmatrix} 2 & 3 & 2 & 3 \\ 0 & -11 & 1 & -13 \\ 0 & 0 & \frac{34}{11} & -\frac{156}{11} \end{pmatrix}$$

$$\begin{cases} 2x+3y+2z=3 \\ -11y+z=-13 \\ \frac{34}{11}z=-\frac{156}{11} \end{cases} \Longrightarrow \begin{cases} x=\frac{84}{17} \\ y=\frac{13}{17} \\ z=-\frac{78}{17} \end{cases}$$

2. $\begin{pmatrix} 2 & 3 & 2 & 1 \\ 1 & 0 & 3 & -7 \\ 2 & 2 & 3 & 3 \end{pmatrix} \xrightarrow{2 \Leftrightarrow 1} \begin{pmatrix} 1 & 0 & 3 & -7 \\ 2 & 3 & 2 & 1 \\ 2 & 2 & 3 & 3 \end{pmatrix}$

$\xrightarrow[3-1]{2-1} \begin{pmatrix} 1 & 0 & 3 & -7 \\ 0 & 3 & -4 & 15 \\ 0 & 2 & -3 & 17 \end{pmatrix} \xrightarrow{3 - \frac{2}{3} 2} \begin{pmatrix} 1 & 0 & 3 & -7 \\ 0 & 3 & -4 & 15 \\ 0 & 0 & -\frac{1}{3} & 7. \end{pmatrix}$

$\begin{cases} x + 3z = -7 \\ 3y - 4z = 15 \\ -\frac{1}{3} z = 7 \end{cases} \Longrightarrow \begin{cases} x = 56. \\ y = -23 \\ z = -21 \end{cases}$

Problem 8. $\begin{pmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix} \xrightarrow{2 \Leftrightarrow 1} \begin{pmatrix} 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix} \xrightarrow[3-1]{2-k1}$

$\longrightarrow \begin{pmatrix} 1 & k & 1 & 1 \\ 0 & 1-k^2 & 1-k & 1-k \\ 0 & 1-k & k-1 & 0 \end{pmatrix}$ $(*)$

Case 1. $k=1$ $\quad (*) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow x+y+z=1$

(infinitely many solutions)

Case 2 $k \neq 1$ $(*) \xrightarrow[\frac{1}{1-k} \times 3]{\frac{1}{1-k} \times 2} \begin{pmatrix} 1 & k & 1 & 1 \\ 0 & 1+k & 1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{2 \Leftrightarrow 3} \begin{pmatrix} 1 & k & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1+k & 1 & 1 \end{pmatrix} \xrightarrow{3-(1+k)2}$

$$\begin{pmatrix} 1 & k & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k+2 & 1 \end{pmatrix} \longrightarrow \begin{cases} x + ky + z = 1 \\ y - z = 0 \\ (k+2)z = 1 \end{cases}$$

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If $k = -2$, there is no solution

Otherwise. $z = \frac{1}{k+2} = y$ $\quad x = \frac{1}{k+2}$ (unique solution)

Problem 9. 1. $e^A = e^{I + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} = I \cdot e^{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

2. $e^A = \begin{pmatrix} 1 & a & b + \frac{ac}{2} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

3. Consider $x' = \begin{pmatrix} 3 & -10 \\ 1 & -4 \end{pmatrix} x$. $\quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -10 \\ 1 & -4-\lambda \end{vmatrix} = 0$

$$\Longleftrightarrow (\lambda+2)(\lambda-1) = 0$$

$\lambda_1 = -2$ $\quad \begin{pmatrix} 5 & -10 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 1$ $\quad \begin{pmatrix} 2 & -10 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$$\Phi(t) = \begin{pmatrix} 2e^{-2t} & 5e^{t} \\ e^{-2t} & e^{t} \end{pmatrix} \qquad \Phi(0) = \begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix} \qquad \Phi^{-1}(0) = \frac{1}{-3}\begin{pmatrix} 1 & -1 \\ -5 & 2 \end{pmatrix}$$

$$e^{A} = \Phi(\cancel{t}) \cdot \Phi^{-1}(0) = -\frac{1}{3} \begin{pmatrix} 2e^{-2} & 5e \\ e^{-2} & e \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -5 & 2 \end{pmatrix}$$

$$= -\frac{1}{3} \begin{pmatrix} 2e^{-2} - 25e & -2e^{-2} + 10e \\ e^{-2} - 5e & -e^{-2} + 2e \end{pmatrix}$$

4. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^2 = I \quad A^3 = A$ - - - - -

$A^{2n} = I$ and $A^{2n+1} = A$.

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{I}{(2k)!} + \sum_{k=0}^{\infty} \frac{A}{(2k+1)!}$$

since $e^{x} \cancel{\cos 1} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $e^{-x} = \sum_{k=0}^{\infty} \frac{x^k (-1)^k}{k!}$

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{e^{\cancel{1}} + e^{-\cancel{1}}}{2} = \cosh \cancel{1} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{e^{\cancel{1}} - e^{-\cancel{1}}}{2} = \sinh \cancel{1}$$

So $e^{A} = \cancel{\cosh I +} (\cosh 1) I + (\sinh 1) A$

$$= \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$$

Problem 10

1. $A=\begin{pmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{pmatrix}$ $\quad \det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 9 & -1-\lambda & 2 \\ -9 & 4 & -1-\lambda \end{vmatrix} = (3-\lambda)\left((\lambda+1)^2+1 \right)$

$\lambda_1 = 3.$ $\quad \begin{pmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} z=0 \\ 9x-4y+2z=0 \\ -9x+4y-4z=0 \end{cases} \Rightarrow \begin{cases} 9x-4y=0 \\ z=0 \end{cases}$

$v=\begin{pmatrix} 4 \\ 9 \\ 0 \end{pmatrix}$ $\quad x_1 = e^{3t}\begin{pmatrix} 4 \\ 9 \\ 0 \end{pmatrix}$

$\lambda_2 = -1+i$ $\quad \begin{pmatrix} 4-i & 0 & 1 \\ 9 & -i & 2 \\ -9 & 4 & -i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} (4-i)x+z=0 \\ 9x-iy+2z=0 \\ -9x+4y-iz=0 \end{cases}$

$v=\begin{pmatrix} 1 \\ 2-i \\ i-4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\tilde{x}_2 = e^{(-1+i)t}\left(\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right) = e^{-t}(\cos t + i\sin t)\left(\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right)$

$= e^{-t}\left(\cos t\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right) + i\left(e^{-t}\sin t\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + e^{-t}\cos t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right)$

$$X_2(t) = e^{-t}\begin{pmatrix} \cos t - \sin t \\ 2\cos t + \sin t \\ -4\cos t - \sin t \end{pmatrix} \qquad X_3(t) = e^{-t}\begin{pmatrix} \sin t \\ 2\sin t - \cos t \\ -4\sin t + \cos t \end{pmatrix}$$

$$X = C_1X_1(t) + C_2X_2(t) + C_3X_3(t)$$

2. $tA = 2tI + B \qquad B = \begin{pmatrix} 0 & t & 0 & t \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 0 & 0 & t^2 & 0 \\ 0 & 0 & 0 & t^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & t^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B^n = 0 \quad (n \geq 4)$$

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{tA} = e^{2t}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad X(t) = e^{tA}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}.$$

Problem 11

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad At = tI + B$$

where $B = \begin{pmatrix} 0 & 2t & 3t & 4t \\ 0 & 0 & 6t & 3t \\ 0 & 0 & 0 & 2t \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad B^2 = \begin{pmatrix} 0 & 0 & 12t^2 & 12t^2 \\ 0 & 0 & 0 & 12t^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad B^3 = \begin{pmatrix} 0 & 0 & 0 & 24t^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$B^n = 0$ for $n \geq 4$

$$e^B = \begin{pmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad e^{At} = e^t \begin{pmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{-B} = \begin{pmatrix} 1 & -2t & -3t+6t^2 & -4t+6t^2-24t^3 \\ 0 & 1 & -6t & -3t+6t^2 \\ 0 & 0 & 1 & -2t \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad e^{-At} = e^{-t} \begin{pmatrix} 1 & -2t & -3t+6t^2 & -4t+6t^2-24t^3 \\ 0 & 1 & -6t & -3t+6t^2 \\ 0 & 0 & 1 & -2t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x_p = e^{tA} \int e^{-tA} e^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} dt = e^{tA} \int \begin{pmatrix} -4t+6t^2-24t^3 \\ -3t+6t^2 \\ -2t \\ 1 \end{pmatrix} dt$$

$$= e^{tA} \begin{pmatrix} -2t^2+2t^3-6t^4 \\ -\frac{3t^2}{2}+2t^3 \\ -t^2 \\ t \end{pmatrix} =$$

$$= e^{t}\begin{pmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2t^2+2t^3-6t^4 \\ -\frac{3t^2}{2}+2t^3 \\ -t^2 \\ t \end{pmatrix}$$ (14)

$$= e^{t}\begin{pmatrix} -2t^2+2t^3-6t^4-3t^3+4t^4-3t^3-6t^4+4t^2+6t^3+4t^4 \\ -\frac{3t^2}{2}+2t^3-6t^3+3t^2+6t^3 \\ -t^2+2t^2 \\ t \end{pmatrix}$$

$$= e^{t}\begin{pmatrix} -4t^4+2t^3+2t^2 \\ 2t^3-\frac{3}{2}t^2 \\ t^2 \\ t \end{pmatrix}$$

$$X = X_p + X_c = X_p + e^{-t}\begin{pmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

Since $X(0)=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$