Practice Final Exam

Problem 1. Find the general solution of the following differential equations.

1. $yy' = x(1 + y^2)$
2. $(1 + x)y' + y = \cos x$.
3. $x(x + y)y' + y(3x + y) = 0$.
4. $y' = \sqrt{x + y + 2}$.
5. $xy' + 6y = 3xy^{\frac{4}{3}}$.
6. $(\cos x + \ln y)\, dx + (\frac{x}{y} + e^y)\, dy = 0$.

Problem 2. A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

Problem 3. Find the general solution of the following higher-order differential equations.

1. $y'' - 3y' + 2y = 0$.
2. $4y'' + 4y' + y = 0$.
3. $y'' + 6y' + 10y = 0$.
4. $y^{(3)} + 2y'' - y' - 2y = 0$.
5. $y^{(3)} + 3y'' + 3y + y = 0$.

Problem 4. Solve the following initial value problems.

1. $y^{(3)} = y$; $y(0) = 1$, $y'(0) = y''(0) = 0$.
2. $y'' + 2y' + 2y = e^{-x}$; $y(0) = 1$, $y'(0) = 2$.

Problem 5. Let A and B be two 2×2 matrice. Prove that $\det(AB) = \det(A) \cdot \det(B)$.

Problem 6. Let A and B be two $n \times n$ matrice. Prove that $(AB)^T = B^T A^T$.

Problem 7. Solve the following systems of linear equations.

1. $\begin{cases} 2x+3y+2z=3 \\ 4x-5y+5z=-7 \\ -3x+7y-2z=5 \end{cases}.$

2. $\begin{cases} 2x+3y+2z=1 \\ x+0y+3z=-7 \\ 2x+2y+3z=3 \end{cases}.$

Problem 8. Consider the following system of linear equations

$$\begin{cases} kx+y+z=1 \\ x+ky+z=1 \\ x+y+kz=1 \end{cases}.$$

For what value(s) of k does this have (i) a unique solution? (ii) no solution? (iii) infinitely many solutions?

Problem 9. For the matrix A given below, compute $\exp(A)$.

1. $A=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$

2. $A=\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ for some constants a, b, c.

3. $A=\begin{pmatrix} 3 & -10 \\ 1 & -4 \end{pmatrix}.$

4. $A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

Problem 10. Solve the following homogeneous systems.

1. $\begin{cases} x'=3x+z \\ y'=9x-y+2z \\ z'=-9x+4y-z \end{cases}.$

2. $\mathbf{x}'=\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}\mathbf{x}.$

Problem 11. Solve the following initial value problem.

$$\mathbf{x}'=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\mathbf{x}+e^t\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}(0)=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$