Part III

Problem 9

$V_1 = 50$ gal $\quad V_2 = 25$ gal.

$r = 10$ gal/min $\quad x_1(0) = 15$

$x_2(0) = 0.$

$$\begin{cases} \dfrac{dx_1}{dt} = -k_1 x_1 + k_2 x_2 \\ \dfrac{dx_2}{dt} = k_1 x_1 - k_2 x_2 \end{cases}$$

$$k_1 = \frac{r}{V_1} = \frac{10}{50} = \frac{1}{5} \qquad k_2 = \frac{r}{V_2} = \frac{10}{25} = \frac{2}{5}$$

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{1}{5} x_1 + \dfrac{2}{5} x_2 \\ \dfrac{dx_2}{dt} = \dfrac{1}{5} x_1 - \dfrac{2}{5} x_2 \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} -1/5 & 2/5 \\ 1/5 & -2/5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} -1/5 - \lambda & 2/5 \\ \frac{1}{5} & -\frac{2}{5} - \lambda \end{pmatrix} =$$

$$= \left(-\frac{1}{5} - \lambda\right)\left(-\frac{2}{5} - \lambda\right) - \frac{2}{5}\frac{1}{5} =$$

$$= \frac{2}{25} + \frac{3}{5}\lambda + \lambda^2 - \frac{2}{25} = 0$$

$$25\lambda^2 + 15\lambda = 0$$

$$5\lambda(5\lambda + 3) = 0$$

$$\boxed{\lambda_1 = 0 \qquad \lambda_2 = -3/5 \qquad \text{both alg. multipl. } 1}$$

Eigenvector for $\lambda_1 = 0$.

Plug $\lambda = 0$, look for $\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that

$$\begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -2/5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \cancel{-\frac{1}{5}a + \frac{2}{5}b = 0} \\ \frac{1}{5}a - \frac{2}{5}b = 0 \end{cases} \rightarrow \quad a = 2b$$

b is free

We get one eigenvector $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

and the solution $\boxed{\underline{x}_1(t) = \underline{v}_1 e^{0t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

Eigenvector for $\lambda_2 = -\frac{3}{5}$

Plug $\lambda_2 = -3/5$ and look for a vector $\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that

$$\begin{pmatrix} 2/5 & 2/5 \\ 1/5 & 1/5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \cancel{2/5\, a + 2/5\, b} = 0 \\ 1/5\, a + 1/5\, b = 0 \quad \rightarrow \quad a = -b \end{cases}$$

b is free

We get one eigenvector $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

and the solution

$$\underline{x}_2(t) = \underline{v}_2\, e^{\lambda_2 t} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3/5\, t}$$

The general solution is

$$\underline{x} = c_1 \underline{x}_1 + c_2 \underline{x}_2 = \begin{pmatrix} 2c_1 - c_2 e^{-3/5 t} \\ c_1 + c_2 e^{-3/5 t} \end{pmatrix}$$

Initial conditions: (plug $t=0$)

$$\begin{cases} 2c_1 - c_2 = 15 \quad \rightarrow \quad c_2 = 2c_1 - 15 = -2c_2 - 15 \\ c_1 + c_2 = 0 \quad \rightarrow \quad c_1 = -c_2 \end{cases}$$

$$3c_2 = -15 \quad \longrightarrow \quad \boxed{c_2 = -15/3}$$

$$\boxed{c_1 = -c_2 = 15/3}$$

The particular solution is

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 + \frac{15}{3} e^{-3/5 t} \\ 15/3 - \frac{15}{3} e^{-3/5 t} \end{pmatrix}}$$

Problem 10

$$\begin{cases} x_1' = 5x_1 \\ x_2' = 10x_1 + 5x_2 \\ x_3' = 20x_1 + 30x_2 + 5x_3 \end{cases} \qquad \begin{cases} x_1(0) = 4 \\ x_2(0) = 5 \\ x_3(0) = 6 \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} 5 & 0 & 0 \\ 10 & 5 & 0 \\ 20 & 30 & 5 \end{pmatrix}}_{\overset{\shortparallel}{A}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}}_{\overset{\shortparallel}{D}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{pmatrix}}_{\overset{\shortparallel}{B}}.$$

Want to find e^{At}.

Because $BD = DB$, we have

$$e^{At} = e^{(D+B)t} = e^{Dt}\, e^{Bt}$$

$$e^{Dt} = \begin{pmatrix} e^{5t} & 0 & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}$$ because D is diagonal.

Now we find e^{Bt}.

$$B^2 = \begin{pmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 300 & 0 & 0 \end{pmatrix}$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 300 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^{Bt} = I + Bt + \frac{1}{2} B^2 t^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 10t & 0 & 0 \\ 20t & 30t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 150t^2 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 10t & 1 & 0 \\ 150t^2 + 20t & 30t & 1 \end{pmatrix}$$

Hence

$$e^{At} = \begin{pmatrix} e^{5t} & 0 & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 10t & 1 & 0 \\ 150t^2 + 20t & 30t & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} e^{5t} & 0 & 0 \\ 10t\, e^{5t} & e^{5t} & 0 \\ (150t^2 + 20t) \cdot e^{5t} & (30t) \cdot (e^{5t}) & e^{5t} \end{pmatrix}$$

b. Particular solution of the system is

$$\underline{x}(t) = e^{At} \cdot \underline{x}_0 =$$

$$= \begin{pmatrix} e^{5t} & 0 & 0 \\ 10t\, e^{5t} & e^{5t} & 0 \\ (150t^2 + 20t) \cdot e^{5t} & (30t) \cdot e^{5t} & e^{5t} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} =$$

$$= \begin{pmatrix} 4\, e^{5t} \\ (40t + 5)\, e^{5t} \\ [4(150t^2 + 20t) + 150t + 6]\, e^{5t} \end{pmatrix}$$