Final Exam — May 21, 2013, 8:00 to 10:45 AM

Name: Solution Key

Circle your recitation:

R01 (Claudio · Fri) R02 (Xuan · Wed) R03 (Claudio · Mon)

- **You have a maximum of $2\frac{1}{2}$ hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.
- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

Grading

1	/15	**6**	/10
2	/15	**7**	/20
3	/20	**8**	/10
4	/10	**9**	/20
5	/15	**10**	/15
		Total	/150

1. (*15 points*) Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1-2x}{y}$.

(a) (*10 points*) Find the general solution to this DE.

Solution: This DE is separable, so we write it as $y\frac{dy}{dx} = 1 - 2x$. Integrating, we obtain

$$\frac{1}{2}y^2 = x - x^2 + C \quad \Rightarrow \quad y^2 = 2x - 2x^2 + C \quad \Rightarrow \quad y = \pm\sqrt{2x - 2x^2 + C}.$$

Note that because the square-root function always returns non-negative values, we require a $\pm$ sign to capture all of the solutions.

(b) (*5 points*) Find the solution matching the condition $y(1) = -2$. On what interval is this solution defined?

Solution: We apply $x = 1$ and $y = -2$ to the implicit form of the solution to solve for C. Then $4 = 2(1) - 2(1)^2 + C$, so $C = 4$. Since $y(1) < 0$, we must take the negative branch of the solution, so

$$y(x) = -\sqrt{2x - 2x^2 + 4}.$$

We note that this function is defined where $2x - 2x^2 + 4 \geq 0$. Equivalently, $x^2 - x - 2 \leq 0$, so since $x^2 - x - 2 = (x-2)(x+1)$, this is where $-1 \leq x \leq 2$. From the original DE, though, we should exclude points where $y = 0$, so the solution is defined on the open interval $(-1, 2)$.

2. (*15 points*) Find the general solution to the linear system

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Solution: Letting the matrix above be denoted A, we find its eigenvalues. As a shortcut, we note that it it lower triangular, so its eigenvalues are the values along the diagonal: 1, with multiplicity 2, and 2. To check, we compute $\det(A - \lambda I)$ by row expansion:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(1-\lambda).$$

We find eigenvectors for these eigenvalues. First, we find solutions $\mathbf{v} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ to $(A - I)\mathbf{v} = \mathbf{0}$ by row reducing $A - I$:

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Translating this back into equations, the two non-trivial rows give that $a = 0$ and $b = 0$, so $\mathbf{v} = \begin{bmatrix} 0 & 0 & c \end{bmatrix}^T$, with c still a free variable. Letting $c = 1$, we obtain a single linearly independent eigenvector, $\mathbf{v}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Since $\lambda = 1$ has multiplicity 2, we expect a generalized eigenvector $\mathbf{v}_2 = \begin{bmatrix} a & b & c \end{bmatrix}^T$ with $(A - I)\mathbf{v}_2 = \mathbf{v}_1$, which we solve by row reducing $[A - I \mid \mathbf{v}_1]$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then $a = 1$ and $b = -1$, so $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 & c \end{bmatrix}^T$. Choosing $c = 0$ this time, $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$ (although any choice of c is valid).

Next, we find an eigenvector $\mathbf{v}_3 = \begin{bmatrix} a & b & c \end{bmatrix}^T$ for $\lambda = 2$ by row reduction of $A - 2I$:

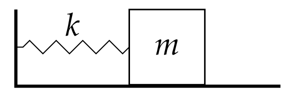
$$A - 21 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $a = 0$ and $2b - c = 0$, so $2b = c$. Taking $b = 1$, $c = 2$, and $\mathbf{v}_3 = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^T$. Therefore, the general solution is

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^t \left(t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) + c_3 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

3. (*20 points*)

An 8 kg mass m is attached to a spring of constant $k = 2$ N/m, and allowed to return to its equilibrium position. The system is naturally damped due to friction with a constant of $c = 8$ N/m-s.



(a) (*5 points*) Find the general solution to the motion $x(t)$ of the mass.

Solution: The DE describing the motion is $8x'' + 8x' + 2x = 0$, which we normalize to $x'' + x' + \frac{1}{4}x = 0$. Its characteristic equation is

$$r^2 + r + \frac{1}{4} = 0 \quad \Rightarrow \quad \left(r + \frac{1}{2}\right)^2 = 0,$$

so it has a double root at $r = -\frac{1}{2}$, and therefore the general solution is

$$x(t) = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t}.$$

(b) (*5 points*) Suppose that the mass is moved 10 cm to the right of the equilibrium position and released at a speed of 7 cm/s to the left at time $t = 0$. Find the displacement $x(t)$ of the mass, in cm.

Solution: From the statement, the initial conditions are $x(0) = 10$ and $x'(0) = -7$, where x is measured in cm. Since $x'(t) = -\frac{1}{2}c_1 e^{-\frac{1}{2}t} + c_2(-\frac{1}{2}te^{-\frac{1}{2}t} + e^{-\frac{1}{2}t})$, $x'(0) = -\frac{1}{2}c_1 + c_2$. Then $x(0) = c_1$, so $c_1 = 10$, and $c_2 = -7 + \frac{1}{2}(10) = -7 + 5 = -2$. Hence, the displacement function is

$$x(t) = (10 - 2t)e^{-\frac{1}{2}t}.$$

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(c) (*5 points*) Find the first time $t > 0$ at which the mass crosses the equilibrium position.

Solution: The mass crosses the equilibrium whenever its displacement is 0. Using that $x(t) = (10 - 2t)e^{-\frac{1}{2}t}$, we solve $(10 - 2t)e^{-\frac{1}{2}t} = 0$: dividing by the never-zero exponential, $10 - 2t = 0$, so $2t = 10$, and $t = 5$ seconds.

(d) (*5 points*) An external force $f(t) = 20 \cos t$ is then applied to the mass. Find the amplitude of the steady-state motion of the mass.

Solution: Since the forcing term contains $\cos t$, and since neither $\cos t$ nor $\sin t$ appears in the general solution of the unforced equation, we guess $x_p(t) = A \cos t + B \sin t$ as a particular solution. We plug its derivatives

$$x_p' = -A \sin t + B \cos t, \qquad x_p'' = -A \cos t - B \sin t = -x_p,$$

into the forced DE, $8x'' + 8x' + 2x = 20 \cos t$:

$$8x_p'' + 8x_p' + 2x_p = 8x_p' - 6x_p = -8A \sin t + 8B \cos t - 6A \cos t - 6B \sin t = 20 \cos t.$$

Isolating the coefficients of the $\cos t$ and $\sin t$ components, we have the linear system $-6A + 8B = 20$ and $-8A - 6B = 0$. Then $B = -\frac{4}{3}A$, so $-6A - \frac{32}{3}A = 20$, and $-50A = 60$. Hence, $A = -\frac{6}{5}$, so $B = \frac{8}{5}$. Then the amplitude is

$$C = \sqrt{A^2 + B^2} = \sqrt{\left(-\frac{6}{5}\right)^2 + \left(\frac{8}{5}\right)^2} = \sqrt{\frac{10^2}{5^2}} = 2.$$

4. (*10 points*) Fresh water flows at a constant rate of $r = 6$ liters/minute into a tank contaning $V_1 = 20$ liters of salt solution. The well-mixed solution then flows at the same rate into a second tank containing $V_2 = 30$ liters of solution, and then drains out of that tank at the same rate.

(a) (*5 points*) Write a linear system of DEs describing the amount of salt $x_1(t)$ and $x_2(t)$ in each of the two tanks.

Solution: For each tank, the rate of change of the amount of salt is $r_{in}c_{in} - r_{out}c_{out}$. For each tank, $r_{in} = r_{out} = 6$. For tank 1, $c_{in} = 0$ and $c_{out} = \frac{x_1}{V_1} = \frac{1}{20}x_1$, so

$$x_1' = 0 - (6)\frac{1}{20}x_1 = -\frac{3}{10}x_1.$$

For tank 2, $c_{in} = \frac{x_1}{V_1} = \frac{1}{20}x_1$, while $c_{out} = \frac{x_2}{V_2} = \frac{1}{30}x_2$, so

$$x_2' = (6)\frac{1}{20}x_1 - (6)\frac{1}{30}x_2 = \frac{3}{10}x_1 - \frac{1}{5}x_2.$$

(b) (*5 points*) The general solution to this system is $x_1(t) = c_1e^{-3/10t}$, $x_2(t) = c_2e^{-1/5t} - 3c_1e^{-3/10t}$. At $t = 0$, tank 1 contains salt at a concentration of 0.5 kg/liter and tank 2 at a concentration of 0.2 kg/liter. Find $x_1(t)$ and $x_2(t)$ matching this initial condition.

Solution: Since we are given the initial concentrations, not the amounts, we multiply by the tank volumes to obtain $x_1(0) = (0.5)(20) = 10$ and $x_2(0) = (0.2)(30) = 6$. Then $x_1(0) = c_1 = 10$ and $x_2(0) = c_2 - 3c_1 = 6$, so $c_2 = 6 + 3c_1 = 36$. Hence, the solution with this IC is

$$x_1 = 10e^{-3/10t}, \qquad x_2(t) = 36e^{-1/5t} - 30e^{-3/10t}.$$

5. (*15 points*) Consider the linear system $x' = 3x - 5y$, $y' = x - y$.

(a) (*10 points*) Find the general solution to this system.

Solution: Writing the system as $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$, we find the eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 5 = \lambda^2 - 2\lambda + 2.$$

The roots of this characteristic polynomial are $\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. We find a complex eigenvector for $\lambda = 1 - i$ by row reduction of $A - (1-i)I$:

$$A - (1-i)I = \begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 1 & -2+i \end{bmatrix} \sim \begin{bmatrix} 1 & -2+i \\ 0 & 0 \end{bmatrix}.$$

Then $\mathbf{v}_1 = \begin{bmatrix} 2-i & 1 \end{bmatrix}^T$ is a reasonable complex eigenvector, giving the complex-valued solution

$$\begin{aligned} \mathbf{x}_1(t) &= e^{(1-i)t}\mathbf{v}_1 = e^t(\cos t - i \sin t) \begin{bmatrix} 2-i \\ 1 \end{bmatrix} \\ &= e^t \left(\begin{bmatrix} 2\cos t + \sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} -2\sin t - 2\cos t \\ -\sin t \end{bmatrix} \right). \end{aligned}$$

Taking linear combinations of the real and imaginary parts of this solution (and making the choice to multiply the imaginary part by -1, to remove the negative signs) gives the general real-valued solution:

$$x(t) = c_1 e^t \begin{bmatrix} 2\cos t + \sin t \\ \cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} 2\sin t + 2\cos t \\ \sin t \end{bmatrix}.$$

(b) (*5 points*) Characterize the behavior of the system around its only critical point, $(0, 0)$.

Solution: Since A has complex eigenvalues, $\lambda = 1 \pm i$, with positive real part, we expect an unstable spiral point at the origin.

6. (*10 points*) Find the general solution to the differential equation $x\dfrac{dy}{dx} = 2y + x^3 \cos x$.

Solution: We normalize the DE to isolate $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{2}{x}y + x^2 \cos x$. We then recognize the DE as linear and rearrange it into the normal form

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x.$$

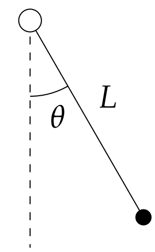
The corresponding integrating factor is then $\mu(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln|x|} = x^{-2}$. Multiplying the DE by $\mu(x)$, it becomes

$$(x^{-2}y)' = x^{-2}x^2 \cos x = \cos x,$$

which we integrate to obtain $x^{-2}y = \sin x + C$. Isolating y by multiplying by x^2, the general solution is

$$y = x^2 \sin x + Cx^2.$$

7. (*20 points*)



Consider a damped pendulum of length $L = 5/8$ m. Assuming that $g = 10$ m/s^2, the angle $\theta(t)$ it makes with the vertical is controlled by the nonlinear differential equation $\theta'' + 6\theta' + 16\sin\theta = 0$. Introducing the new variable $\omega = \theta'$, we obtain the nonlinear autonomous system

$$\theta' = \omega, \qquad \omega' = -16\sin\theta - 6\omega.$$

(a) (*5 points*) Find all of the critical points (θ, ω) of this system with $0 \leq \theta < 2\pi$.

Solution: To find the critical points, we must find the values in this range where $\theta' = 0$ and $\omega' = 0$ simultaneously. From the first equation, $\omega = 0$. From the second, $-16\sin\theta - 6\omega = 0$, so $\sin\theta = 0$ as well. This occurs where $\theta = n\pi$ for all integers n, so in this range the values of θ are 0 and π. Hence, the critical points (θ, ω) are $(0, 0)$ and $(\pi, 0)$.

(b) (*5 points*) Find the Jacobian matrix $J(\theta, \omega)$ of this system.

Solution: Letting $\theta' = F(\theta, \omega) = \omega$ and $\omega' = G(\theta, \omega) = -16\sin\theta - 6\omega$, the Jacobian matrix is

$$J(\theta, \omega) = \begin{bmatrix} F_\theta(\theta, \omega) & F_\omega(\theta, \omega) \\ G_\theta(\theta, \omega) & G_\omega(\theta, \omega) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -16\cos\theta & -6 \end{bmatrix}.$$

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(c) (*10 points*) Characterize the behavior of the system at the critical points you found in part (a). Interpret this behavior in terms of the motion of the pendulum.

Solution: We evaluate the Jacobian matrix at these two critical points:

- At $(0,0)$, $\cos\theta = 1$, so $J(0,0) = \begin{bmatrix} 0 & 1 \\ -16 & -6 \end{bmatrix}$. Computing its eigenvalues,

$$\det J(0,0) - \lambda I = \begin{vmatrix} -\lambda & 1 \\ -16 & -6-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 16.$$

  This polynomial has roots $\lambda = \frac{-6\pm\sqrt{36-64}}{2} = -3 \pm i\sqrt{7}$, which are complex with negative real part. Hence, the system has an asymptotically stable spiral point at $(0,0)$.

- At $(\pi,0)$, $\cos\theta = -1$, so $J(\pi,0) = \begin{bmatrix} 0 & 1 \\ 16 & -6 \end{bmatrix}$. Computing its eigenvalues,

$$\det J(\pi,0) - \lambda I = \begin{vmatrix} -\lambda & 1 \\ 16 & -6-\lambda \end{vmatrix} = \lambda^2 + 6\lambda - 16 = (\lambda+8)(\lambda-2).$$

  Therefore, the eigenvalues are 2 and -8, so the system has a saddle point at $(\pi,0)$, which is always unstable.

In terms of the motion of the pendulum, we see that if the pendulum starts close to $\theta = 0$ (pointing straight downward) with a small angular velocity ω, it will get closer and closer to resting at $\theta = 0$. On the other hand, if it is close to the straight-upward position $\theta = \pi$, it will not settle there, and will instead settle at another equilibrium. There is one trajectory that does converge to this upwards equilibrium, but in practice small forces on the pendulum will perturb it out of this trajectory.

8. (*10 points*) Let $B = \begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix}$. A fundamental matrix for $\mathbf{x}' = B\mathbf{x}$ is $\Phi(t) = \begin{bmatrix} e^t & e^{-2t} \\ 2e^t & e^{-2t} \end{bmatrix}$. Find a particular solution to the system $\mathbf{x}' = B\mathbf{x} + \mathbf{f}(t)$, where $\mathbf{f}(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$.

Solution: From the fundamental matrix given in the problem statement, we see that the eigenvalues of B are 1 and -2, which do not overlap with the eigenvalue 2 represented in the forcing functions. Hence, we may guess a particular solution of the form $\mathbf{x}_p(t) = e^{2t}\mathbf{a}$, where $\mathbf{a} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}^T$. Then $\mathbf{x}_p' = 2e^{2t}\mathbf{a}$. Rearranging the DE to be $\mathbf{x}' - B\mathbf{x} = \mathbf{f}(t)$ and plugging in this guess, we obtain

$$2e^{2t}\mathbf{a} - e^{2t}B\mathbf{a} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \Rightarrow \quad (2I - B)\mathbf{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

where we factor out the $\mathbf{a}$ on the left and divide by the nonzero factor e^{2t}. Then $\mathbf{a} = (2I - B)^{-1}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Since $2I - B = \begin{bmatrix} 7 & -3 \\ 6 & -2 \end{bmatrix}$, and using the positional formula for the inverse of a 2×2 matrix,

$$(2I - B)^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix} \qquad (2I - B)^{-1}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Therefore, a particular solution is $\mathbf{x}_p = \begin{bmatrix} 0 \\ -e^{2t} \end{bmatrix}$.

We can also obtain a particular solution with the variation of parameters formula $\mathbf{x}(t) = \Phi(t)\int \Phi(t)^{-1}\mathbf{f}(t)\,dt$:

$$\begin{aligned}
\Phi(t)^{-1} &= \frac{1}{-e^{-t}}\begin{bmatrix} e^{-2t} & -e^{-2t} \\ -2e^t & e^t \end{bmatrix} = \begin{bmatrix} -e^{-t} & e^{-t} \\ 2e^{2t} & -e^{2t} \end{bmatrix} \\
\Phi(t)^{-1}\mathbf{f}(t) &= \begin{bmatrix} -e^{-t} & e^{-t} \\ 2e^{2t} & -e^{2t} \end{bmatrix}\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} -e^t \\ 4e^{4t} \end{bmatrix} \\
\Phi(t)\int \Phi(t)^{-1}\mathbf{f}(t)\,dt &= \begin{bmatrix} e^t & e^{-2t} \\ 2e^t & e^{-2t} \end{bmatrix}\begin{bmatrix} -e^t \\ e^{4t} \end{bmatrix} = \begin{bmatrix} 0 \\ -e^{2t} \end{bmatrix}.
\end{aligned}$$

This agrees with the answer calculated with undetermined coefficients. This computation could also be done using e^{Bt} as the fundamental matrix, but the additional work required to compute and to work with e^{Bt} is probably not justified, given that it was relatively easy to compute $\Phi(t)^{-1}$ directly.

9. (*20 points*) Ninjas board a ship full of pirates, and they start to fight, thereby reducing each others' populations. Because of the ninjas' superior training and skills, they are four times more effective at fighting than the pirates are. A model governing the evolution of the populations $x(t)$ of pirates and $y(t)$ of ninjas is

$$\frac{dx}{dt} = -2y, \qquad \frac{dy}{dt} = -\frac{1}{2}x.$$

(a) (*5 points*) Find the general solution to this system of DEs.

Solution: We write this system as $\mathbf{x}' = A\mathbf{x}$, with $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} 0 & -2 \\ -\frac{1}{2} & 0 \end{bmatrix}$. We compute the eigenvalues of A:

$$\det(A - \lambda I) = \begin{bmatrix} -\lambda & -2 \\ -\frac{1}{2} & -\lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).$$

Hence, the eigenvalues are 1 and -1. We find eigenvectors for both:

$$A - I = \begin{bmatrix} -1 & -2 \\ -\frac{1}{2} & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

so $\lambda_1 = 1$ has an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Similarly,

$$A + I = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

so $\lambda_2 = -1$ has an eigenvector $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The general solution is then

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

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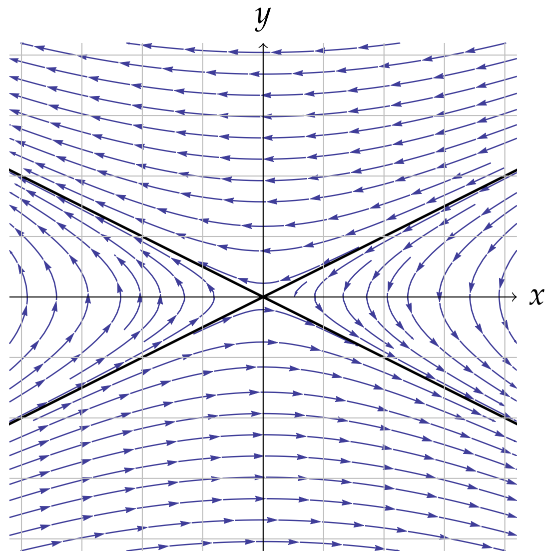
(b) (*5 points*) Plot some trajectories of these populations in the xy-plane on the axes below. Pay close attention to the behavior along any eigendirections.



(c) (*5 points*) Explain what happens in the first quadrant. How can you tell from the starting populations which side will be victorious?

Solution: In the first quadrant, we see that the line $y = \frac{1}{2}x$ separates two regions with different behaviors: below this line, trajectories intersect the positive x-axis, indicating that pirates are left but not ninjas ($x > 0$, but $y = 0$). Conversely, above this line, trajectories intersect the positive y-axis, indicating that ninjas are left but not pirates. Hence, this line is a separatrix between these two outcomes, and the location of the initial condition above or below the separatrix determines the outcome entirely.

(d) (*5 points*) If the pirate ship initially contains 50 pirates, how many ninjas are required to defeat all of them (that is, to reduce their population to 0)?

Solution: In order for all the pirates to be destroyed, at least 25 ninjas are required, so that the trajectory does not intersect the positive x-axis. This differs from the naive estimate that 13 ninjas would be required (rounding up from $50/4 = 12.5$), which one might expect from the 4-to-1 skill advantage. (More generally, an n^2-to-1 skill advantage is required in this model of combat to overcome being outnumbered n-to-1.)

10. (*15 points*) Let $A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$.

(a) (*10 points*) Find e^{At}.

Solution: Although we would like to write $A = D + B$, the sum of a diagonal matrix $D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and a nilpotent matrix $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, this is not useful for computing e^{At}, since D and B do not commute ($BD \neq DB$). Likewise, although $3I$ and $A - 3I$ commute, $A - 3I$ is not nilpotent and therefore does not have an easily computed matrix exponential. Instead, we construct a fundamental matrix $\Phi(t)$ from the eigendata of A. Since A is upper triangular, its eigenvalues are 3 and 4 by inspection. We compute eigenvectors by row reducing $A - 3I$ and $A - 4I$:

$$\begin{aligned} A - 3I &= \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ A - 4I &= \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

Then a fundamental matrix $\Phi(t)$ for the system $\mathbf{x}' = A\mathbf{x}$ is

$$\Phi(t) = \begin{bmatrix} e^{3t} & 2e^{4t} \\ 0 & e^{4t} \end{bmatrix}, \qquad \Phi(0) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \Phi(0)^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

Finally,

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^{3t} & 2e^{4t} \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{3t} & 2e^{4t} - 2e^{3t} \\ 0 & e^{4t} \end{bmatrix}.$$

(b) (*5 points*) Evaluate $\frac{d}{dt}e^{At}$ at $t = 0$.

Solution: We first compute $\frac{d}{dt}e^{At}$: by the properties of the matrix exponential, $\frac{d}{dt}e^{At} = Ae^{At}$. At $t = 0$, this is $Ae^{A(0)} = Ae^0 = AI = A$, so the answer is

$$\left.\frac{d}{dt}e^{At}\right|_{t=0} = A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}.$$

Although we have the answer, we check that it is consistent with our result from part (a). Differentiating e^{At} termwise and evaluating at $t = 0$,

$$\frac{d}{dt}e^{At} = \begin{bmatrix} 3e^{3t} & 8e^{4t} - 6e^{3t} \\ 0 & 4e^{4t} \end{bmatrix}, \qquad \left.\frac{d}{dt}e^{At}\right|_{t=0} = \begin{bmatrix} 3 & 8-6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} = A.$$