Final Exam — May 21, 2013, 8:00 to 10:45 AM

Name: ______________________________

Circle your recitation:

R01 (Claudio · Fri) R02 (Xuan · Wed) R03 (Claudio · Mon)

- **You have a maximum of $2\frac{1}{2}$ hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.
- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

Grading

1	/15	**6**	/10
2	/15	**7**	/20
3	/20	**8**	/10
4	/10	**9**	/20
5	/15	**10**	/15
		Total	/150

1. (*15 points*) Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1-2x}{y}$.

(a) (*10 points*) Find the general solution to this DE.

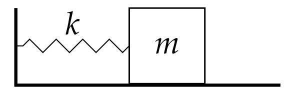
(b) (*5 points*) Find the solution matching the condition $y(1) = -2$. On what interval is this solution defined?

2. (*15 points*) Find the general solution to the linear system

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

3. (*20 points*)

An 8 kg mass m is attached to a spring of constant $k = 2$ N/m, and allowed to return to its equilibrium position. The system is naturally damped due to friction with a constant of $c = 8$ N/m-s.



(a) (*5 points*) Find the general solution to the motion $x(t)$ of the mass.

(b) (*5 points*) Suppose that the mass is moved 10 cm to the right of the equilibrium position and released at a speed of 7 cm/s to the left at time $t = 0$. Find the displacement $x(t)$ of the mass, in cm.

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(c) (*5 points*) Find the first time $t > 0$ at which the mass crosses the equilibrium position.

(d) (*5 points*) An external force $f(t) = 20 \cos t$ is then applied to the mass. Find the amplitude of the steady-state motion of the mass.

4. (*10 points*) Fresh water flows at a constant rate of $r = 6$ liters/minute into a tank contaning $V_1 = 20$ liters of salt solution. The well-mixed solution then flows at the same rate into a second tank containing $V_2 = 30$ liters of solution, and then drains out of that tank at the same rate.

(a) (*5 points*) Write a linear system of DEs describing the amount of salt $x_1(t)$ and $x_2(t)$ in each of the two tanks.

(b) (*5 points*) The general solution to this system is $x_1(t) = c_1 e^{-3/10t}$, $x_2(t) = c_2 e^{-1/5t} - 3c_1 e^{-3/10t}$. At $t = 0$, tank 1 contains salt at a concentration of 0.5 kg/liter and tank 2 at a concentration of 0.2 kg/liter. Find $x_1(t)$ and $x_2(t)$ matching this initial condition.

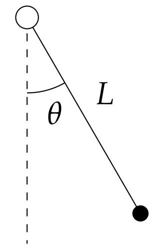
5. (*15 points*) Consider the linear system $x' = 3x - 5y$, $y' = x - y$.

(a) (*10 points*) Find the general solution to this system.

(b) (*5 points*) Characterize the behavior of the system around its only critical point, $(0, 0)$.

6. (*10 points*) Find the general solution to the differential equation $x\dfrac{dy}{dx} = 2y + x^3 \cos x$.

7. (*20 points*)



Consider a damped pendulum of length $L = 5/8$ m. Assuming that $g = 10$ m/s, the angle $\theta(t)$ it makes with the vertical is controlled by the nonlinear differential equation $\theta'' + 6\theta' + 16\sin\theta = 0$. Introducing the new variable $\omega = \theta'$, we obtain the nonlinear autonomous system

$$\theta' = \omega, \qquad \omega' = -16\sin\theta - 6\omega.$$

(a) (*5 points*) Find all of the critical points (θ, ω) of this system with $0 \leq \theta < 2\pi$.

(b) (*5 points*) Find the Jacobian matrix $J(\theta, \omega)$ of this system.

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(c) (*10 points*) Characterize the behavior of the system at the critical points you found in part (a). Interpret this behavior in terms of the motion of the pendulum.

8. (*10 points*) Let $B = \begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix}$. A fundamental matrix for $\mathbf{x}' = B\mathbf{x}$ is $\Phi(x) = \begin{bmatrix} e^t & e^{-2t} \\ 2e^t & e^{-2t} \end{bmatrix}$.

Find a particular solution to the system $\mathbf{x}' = B\mathbf{x} + \mathbf{f}(t)$, where $\mathbf{f}(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$.

9. (*20 points*) Ninjas board a ship full of pirates, and they start to fight, thereby reducing each others' populations. Because of the ninjas' superior training and skills, they are four times more effective at fighting than the pirates are. A model governing the evolution of the populations $x(t)$ of pirates and $y(t)$ of ninjas is

$$\frac{dx}{dt} = -2y, \qquad \frac{dy}{dt} = -\frac{1}{2}x.$$

(a) (*5 points*) Find the general solution to this system of DEs.

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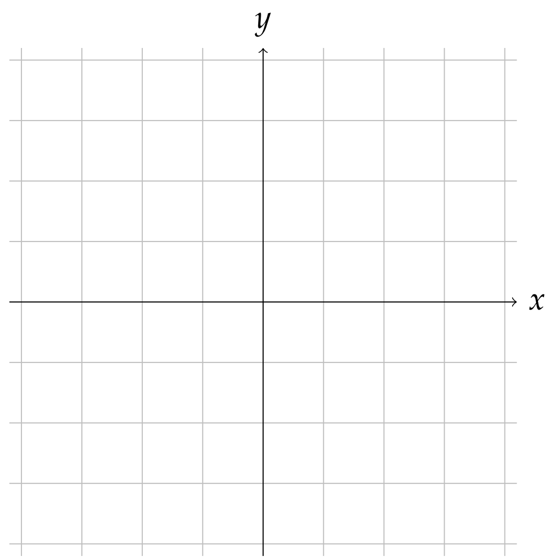
(b) (*5 points*) Plot some trajectories of these populations in the xy-plane on the axes below. Pay close attention to the behavior along any eigendirections.



(c) (*5 points*) Explain what happens in the first quadrant. How can you tell from the starting populations which side will be victorious?

(d) (*5 points*) If the pirate ship initially contains 50 pirates, how many ninjas are required to defeat all of them (that is, to reduce their population to 0)?

10. (*15 points*) Let $A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$.

(a) (*10 points*) Find e^{At}.

(b) (*5 points*) Evaluate $\dfrac{d}{dt}e^{At}$ at $t = 0$.