MAT 132: Calculus 2

Practice Problems: Solutions

Stony Brook University Fall 2021

Problem 1.

$$\int \left(e^x + e^{-x}\right)^2 dx = \int \left(e^{2x} + e^{-2x} + 2\right) dx = \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x} + 2x + C.$$

$$\int_0^{\pi/2} 3\cos^2 x \sin x dx$$

For an integral $\int \cos^n x \sin^m x dx$, where m is odd, we can apply the substitution $u = \cos x, \quad du = -\sin x dx$. We have:

$$\int_0^{\pi/2} 3\cos^2 x \sin x dx = \int_1^0 -3u^2 du = \int_0^1 3u^2 du = u^3\Big|_0^1 = 1.$$

Equivalently, we can first compute the indefinite integral:

$$\int 3\cos^2 x \sin x dx = \int -3u^2 du = -u^3 + C = -\cos^3 x + C,$$

then:

$$\int_0^{\pi/2} 3\cos^2 x \sin x dx = -\cos^3 x\Big|_0^{\pi/2} = 1.$$

$$\int_0^{\pi} \left(\cos^2 x + \cos^2(2x)\right) dx$$

Recall that $\cos^2 x = \frac{1+\cos(2x)}{2}$. Therefore:

$$\int_0^{\pi} \left(\cos^2 x + \cos^2(2x)\right) dx = \int_0^{\pi} \left(\frac{1+\cos(2x)}{2} + \frac{1+\cos(4x)}{2}\right) dx =$$

$$x\Big|_0^{\pi} + \frac{\sin(2x)}{4}\Bigg|_0^{\pi} + \frac{\sin(4x)}{8}\Bigg|_0^{\pi} = \pi.$$

$$\int_0^1 x^2 e^{x^3}\,dx$$

Substitution: $u = x^3$ and $du = 3x^2 dx$. We have:

$$\int_0^1 x^2 e^{x^3} dx = \int_0^1 \frac{e^u}{3} du = \left.\frac{e^u}{3}\right|_0^1 = \frac{e-1}{3}.$$

$$\int \frac{2-x}{x(x+1)} dx$$

Partial fraction decomposition:

$$\frac{2-x}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

or:

$$2 - x = A(x+1) + Bx$$

Setting $x = 0$, we obtain $2 - 0 = A(1+0) + B0$; i.e., $A = 2$. Setting $x = -1$, we obtain $2 - (-1) = A(1-1) + B(-1)$; i.e., $B = -3$. We have:

$$\frac{2-x}{x(x+1)} = \frac{2}{x} - \frac{3}{x+1}$$

and

$$\int \frac{2-x}{x(x+1)} dx = \int \frac{2}{x} dx - \int \frac{3}{x+1} dx = 2\ln|x| - 3\ln|x+1| + C.$$

$$\int \ln(x^2 + x) dx$$

Observe that

$$\int \ln(x^2+x) dx = \int \ln\big(x(x+1)\big) dx = \int \ln(x) dx + \int \ln(x+1) dx.$$

(We assumed that $x > 0$.)

Recall that $\int \ln(x) dx = x\ln x - x + C$. Indeed, integrating by parts:

$$\int \ln(x) dx = x\ln x - \int x(\ln x)' dx = x\ln x - \int 1 dx = x\ln x - x + C.$$

Substituting $u = x + 1$, we obtain

$$\int \ln(x+1)dx = \int \ln u du = u \ln u - u + C = (x+1)\ln(x+1) - (x+1) + C.$$

Therefore,

$$\int \ln(x)dx + \int \ln(x+1)dx = x\ln x - x + (x+1)\ln(x+1) - (x+1) + C.$$

Problem 2. Let R denote the region in the plane bounded by the 4 curves $x = 0$, $x = \pi$, $y = 0$, and $y = \sin x + 1$.

(a) Compute the area of R.

(b) Compute the volume when R is rotated around the x-axis.

Solution. **(a)** The area is

$$\int_0^\pi (\sin x + 1)dx = (-\cos x + x)|_0^\pi = \pi + 2.$$

(b) The volume is

$$\int_0^\pi \pi(\sin x + 1)^2 dx = \int_0^\pi \pi(\sin^2 x + 2\sin x + 1)dx = \int_0^\pi \pi\left(\frac{1-\cos(2x)}{2} + 2\sin x + 1\right)dx$$

$$= \pi\left(\frac{x - \sin(2x)/2}{2} - 2\cos x + x\right)\Big|_0^\pi = 3\pi^2/2 + 4\pi.$$

□

Problem 3. A particle is moving along the x-axis; its speed at any time $t \geq 0$ is given in terms of t by the formula $t^2 e^t$.

Compute the total distance traveled by the particle during the time interval $0 \leq t \leq 2$.

Solution. The total distance is $\int_0^2 t^2 e^t dt$. The integral is computed using integration by parts:

$$\int t^2 e^t dt = t^2 e^t - \int 2te^t dt = t^2 e^t - 2te^t + \int 2e^t dt = t^2 e^t - 2te^t + 2e^t.$$

Therefore,

$$\int_0^2 t^2 e^t dt = t^2 e^t - 2te^t + \int 2e^t dt = t^2 e^t - 2te^t + 2e^t\Big|_0^2 = 2e^2 - 2.$$

□

Problem 4. For each of the following improper integrals, determine whether it converges or not. If the integral converges, then determine its value.

$$\int_{-1}^{2} \frac{dx}{x^3}$$

$$\int_{0}^{\infty} \frac{x}{x^2+1} dx$$

$$\int_{0}^{\infty} \frac{x}{(x^2+1)^2} dx$$

$$\int_{0}^{\infty} \sin^2 x dx$$

Solution. The integral $\int_{-1}^{2} \frac{dx}{x^3}$ diverges because $\int_{-1}^{2} \frac{dx}{x^3} = \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{2} \frac{dx}{x^3}$ and

$$\int_{-1}^{0} \frac{dx}{x^3} = \lim_{t\to 0^-} \int_{-1}^{t} \frac{dx}{x^3} = \lim_{t\to 0^-} \left. \frac{-1}{2x^2} \right|_{-1}^{t} = \lim_{t\to 0^-} \left(\frac{-1}{2t^2} - \frac{-1}{2(-1)^2} \right)$$

diverges.

Substituting $u = x^2 + 1$, we obtain:

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| = \frac{1}{2} \ln |x^2+1|.$$

Therefore,

$$\int_{0}^{\infty} \frac{x}{x^2+1} dx = \lim_{L\to\infty} \left. \frac{1}{2} \ln |x^2+1| \right|_{0}^{L} = \infty$$

diverges.

Substituting $u = x^2 + 1$, we obtain:

$$\int \frac{x}{(x^2+1)^2} dx = \frac{1}{2} \int \frac{du}{u^2} = \frac{-1}{2u} = \frac{-1}{2(x^2+1)}.$$

Therefore,

$$\int_{0}^{\infty} \frac{x}{x^2+1} dx = \lim_{L\to\infty} \left. \frac{-1}{2(x^2+1)} \right|_{0}^{L} = 1/2.$$

Since
$$\int \sin^2 x dx = \int \frac{1-\cos(2x)}{2} dx = \frac{x-\sin(2x)/2}{2} + C,$$
we have
$$\int_0^\infty \sin^2 x dx = \lim_{L\to\infty} \frac{x-\sin(2x)/2}{2}\Big|_0^L = \lim_{L\to\infty} \frac{L-\sin(2L)/2}{2} = \infty$$
diverges.

□

Problem 5. A spring has a natural length of 10 cm. It takes 1 J to stretch the spring from 10 cm to 15 cm. How much work would it take to stretch the spring from 5 cm to 20 cm?

Solution. Since it takes 1 J to stretch the spring from 10 cm to 15 cm, we have:
$$1 = \int_{10-10}^{15-10} kx dx = \int_0^5 kx dx = \frac{25k}{2};$$
hence $k = \dfrac{2}{25}$. We need
$$\int_{5-10}^{20-10} \frac{2}{25} x dx = \int_{-5}^{10} \frac{2}{25} x dx = \frac{x^2}{25}\Big|_{-5}^{10} = 3 \quad J$$
to to stretch the spring from 5 cm to 20 cm.

Remark: it takes 0 J to stretch the spring from 5 cm to 15 cm. □

Problem 6. Find the limits of the following sequences:

a) $\displaystyle\lim_{n\to\infty} \frac{3-n^2}{n^3-n(n^2-1)}$, **b)** $\displaystyle\lim_{n\to\infty} \frac{e^{1-n}}{1+n}$

c) $\displaystyle\lim_{n\to\infty} \frac{(1+n!)^2}{(1-n!)^2}$, **d)** $\displaystyle\lim_{n\to\infty} \left(\frac{2^n}{1+2^{-n}} - 2^n\right)$

e) $\displaystyle\lim_{n\to\infty} \sqrt{\frac{n+3^n}{3^n+5}}$, **f)** $\displaystyle\lim_{n\to\infty} \frac{5n!}{2^n+1}$.

Solution.

a) $$\lim_{n\to\infty} \frac{3-n^2}{n^3-n(n^2-1)} = \lim_{n\to\infty} \frac{3-n^2}{n} = -\infty \qquad \text{diverges}$$

b) $$\lim_{n\to\infty} \frac{e^{1-n}}{1+n} = \lim_{n\to\infty} \frac{e}{(1+n)e^n} = 0$$

c) $$\lim_{n\to\infty} \frac{(1+n!)^2}{(1-n!)^2} = \lim_{n\to\infty} \frac{\left(\frac{1}{n!}+1\right)^2}{\left(\frac{1}{n!}-1\right)^2} = 1$$

d) $$\lim_{n\to\infty} \left(\frac{2^n}{1+2^{-n}} - 2^n\right) = \lim_{n\to\infty} \frac{2^n - 2^n(1+2^{-n})}{1+2^{-n}} = \lim_{n\to\infty} \frac{-1}{1+2^{-n}} = -1$$

e) $$\lim_{n\to\infty} \sqrt{\frac{n+3^n}{3^n+5}} = \lim_{n\to\infty} \sqrt{\frac{n/3^n+1}{1+5/3^n}} = \sqrt{\frac{0+1}{1+0}} = 1$$

f) $$\lim_{n\to\infty} \frac{5n!}{2^n+1} = \infty \qquad \text{diverges}$$

□

Problem 7. Determine if the following series converge absolutely, converge conditionally, or diverge. **No explanation** is required in this problem.

1) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{0.5-2^n}$$

☑ converges absolutely ☐ converges conditionally ☐ diverges

We can compare $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{0.5-2^n}\right|$ to $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

2) $$\sum_{n=10}^{\infty} \frac{2^n-n^5}{n!}$$

☑ converges absolutely ☐ converges conditionally ☐ diverges

The $n!$ in the denominator dominates.

$$\textbf{3)} \qquad \sum_{n=1}^{\infty} \frac{n^{\pi}+2}{n \ln n + 1}$$

☐ converges absolutely ☐ converges conditionally ☑ diverges

We can compare $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\pi}+2}{n \ln n + 1}$ to $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\pi-1}}{\ln n} = \sum_{n=1}^{\infty} \frac{n^{2.14\ldots}}{\ln n}$. Since the sequence $\displaystyle\frac{n^{2.14\ldots}}{\ln n} = \infty$ diverges, the series diverges as well.

$$\textbf{4)} \qquad \sum_{n=1}^{\infty} (-1)^n n \left(\frac{1}{n} - \frac{1}{n+\sqrt{n}} \right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$$

☐ converges absolutely ☑ converges conditionally ☐ diverges

$$\textbf{5)} \qquad \sum_{n=1}^{\infty} \frac{3^{-n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n\ 3^n}$$

☑ converges absolutely ☐ converges conditionally ☐ diverges

Problem 8. Consider the following Maclaurin series

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots.$$

(a) Write the Maclaurin series for $f(x) = \ln(1+2x)$ and for $g(x) = f'(x)$.
(b) What is the radius of convergence for the series in **(a)**?

Solution.

$$f(x) = \ln(1+2x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^n}{n}x^n =$$

$$2x - \frac{2^2}{2}x^2 + \frac{2^3}{3}x^3 - \frac{2^4}{4}x^4 + \frac{2^5}{5}x^5 - \dots,$$

$$g(x) = f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}2^n x^{n-1} = 2 - 2^2x + 2^3x^2 - \dots$$

The radius of convergence for $f(x)$ and $g(x)$ is $\frac{1}{2}$ – it can be easily computed using the ratio test. □

Problem 9. Consider the following Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(a) Write the Maclaurin series for $f(x) = x\sin(x/5)$ and for $g(x) = \int f(x)dx$.
(b) What is the radius and interval of convergence for the series in **(a)**?

Solution.

$$f(x) = x\sin(x/5) = x\sum_{n=0}^{\infty}(-1)^n \frac{(x/5)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{5^{2n+1}(2n+1)!}.$$

$$g(x) = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{5^{2n+1}(2n+1)!}dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+3}}{(2n+3)5^{2n+1}(2n+1)!} + C.$$

The radius of convergence of $f(x)$ and $g(x)$ is $+\infty$ because of the $(2n+1)!$ in the denominator. The interval of convergence is $I = (-\infty, \infty)$. □

Problem 10. Find the general solutions to the following differential equations

$$\textbf{a)} \qquad \frac{dy}{dt} = 2\cos(2t+1)y$$

$$\textbf{b)} \qquad x^2y' = (x+1)y$$

$$\textbf{c)} \qquad y' = e^{x+y}$$

$$\textbf{d)} \qquad y' = x^2e^y$$

Solve the following initial-value problems with the initial condition $y(0) = 1$

$$\textbf{e)} \qquad y' = y + 1$$

$$\textbf{f)} \qquad y' = xy$$

Solution. These are separable differential equations.

$$\textbf{a)} \qquad \int \frac{dy}{y} = \int 2\cos(2t+1)dt$$

$$\ln|y| = \sin(2t+1) + C$$

$$y = \pm e^{\sin(2t+1)+C} = C_2e^{\sin(2t+1)},$$

where $C_2 = \pm e^C$.

$$\textbf{b)} \qquad \int \frac{dy}{y} = \int \frac{x+1}{x^2}dx$$

Note that $\int \frac{x+1}{x^2}dx = \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx = \ln|x| - 1/x + C$. Therefore,

$$\ln|y| = \ln|x| - 1/x + C$$

$$y = \pm e^{\ln|x| - 1/x + C} = C_2xe^{-1/x}.$$

$$\textbf{c)} \qquad \int e^{-y}dy = \int e^x dx$$

$$-e^{-y} = e^x + C$$

$$y = -\ln(-e^x - C)$$

$$\textbf{d)} \qquad \int e^{-y}dy = \int x^2 dx$$

$$-e^{-y} = x^3/3 + C$$
$$y = -\ln\left(-x^3/3 - C\right)$$

$$\textbf{e)} \qquad \int \frac{dy}{y+1} = \int dx$$
$$\ln|y+1| = x + C$$
$$y = \pm e^{x+C} - 1 = C_2 e^x - 1,$$

where $C_2 = \pm e^C$. The initial condition $y(0) = 1$ implies $1 = C_2 e^0 - 1$; i.e. $C_2 = 2$. The answer:

$$y = 2e^x - 1.$$

$$\textbf{f)} \qquad \int \frac{dy}{y} = \int x dx$$
$$\ln|y| = x^2/2 + C$$
$$y = \pm e^{x^2/2+C} = C_2 e^{x^2/2},$$

where $C_2 = \pm e^C$. The initial condition $y(0) = 1$ implies $1 = C_2$. The answer:

$$y = e^{x^2/2}.$$

□

Problem 11. Match the differential equations with corresponding direction vector fields. **No explanation is required in this problem.**

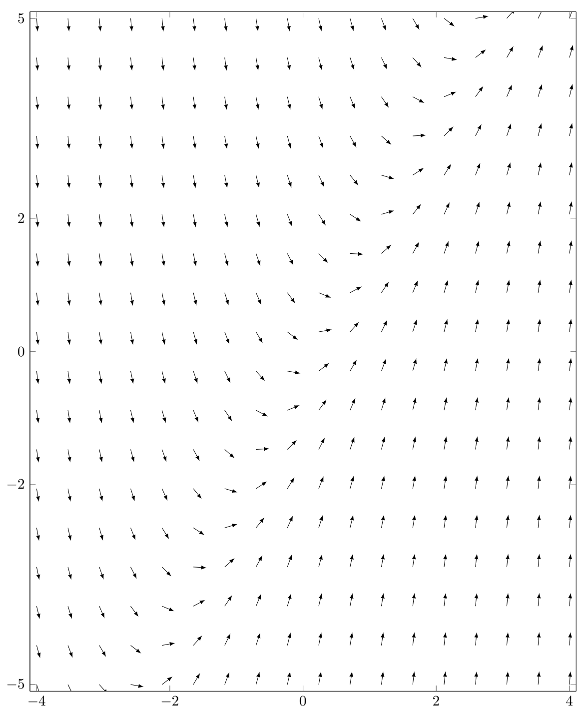
$$y' = x/y, \qquad y' = y(3-y), \qquad y' = x^2 - y^2$$

$$y' = 2x - y, \qquad y' = -2, \qquad y' = 1$$

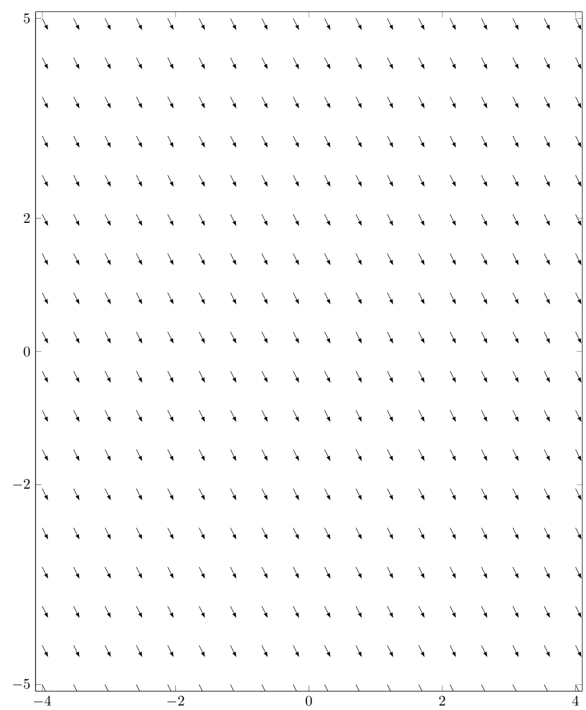
$$y' = \sin x \cos x, \qquad y' = \sin y, \qquad y' = |x|$$

(One equation is without a direction vector field.)

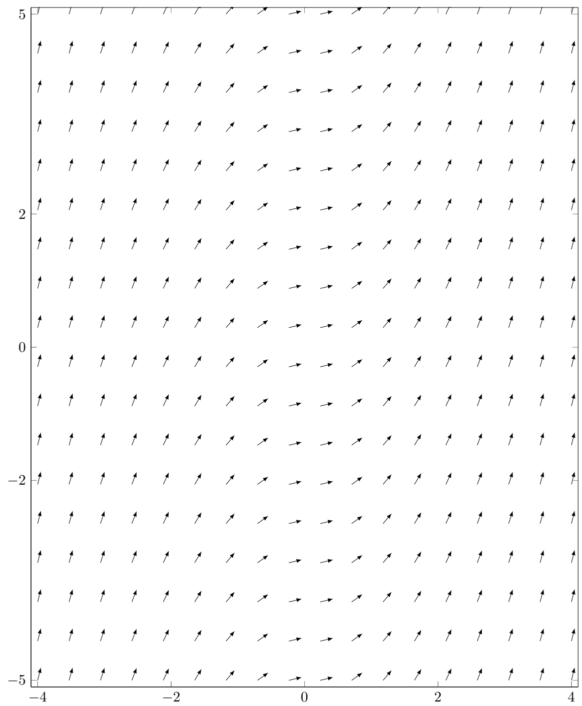
1) $y' = 2x - y$



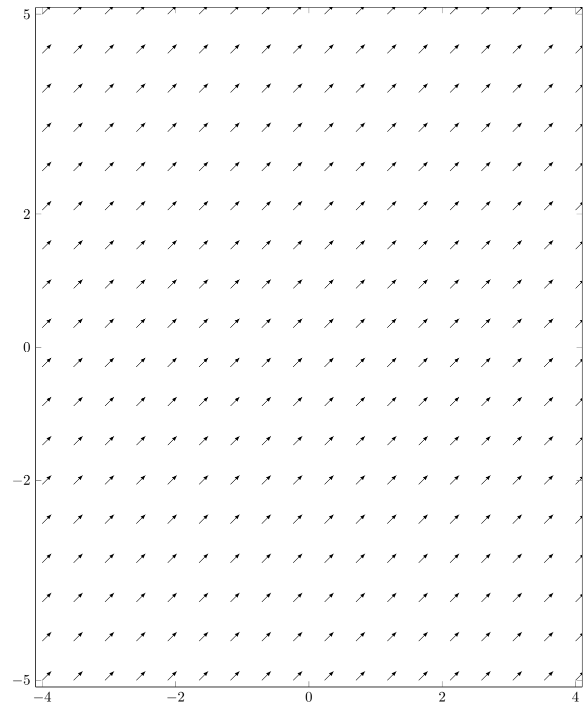
2) $y' = -2$



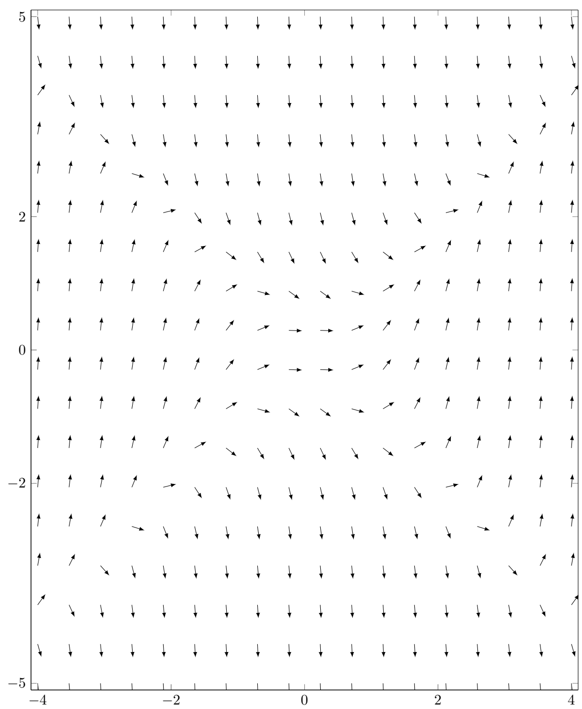
3) $y' = |x|$



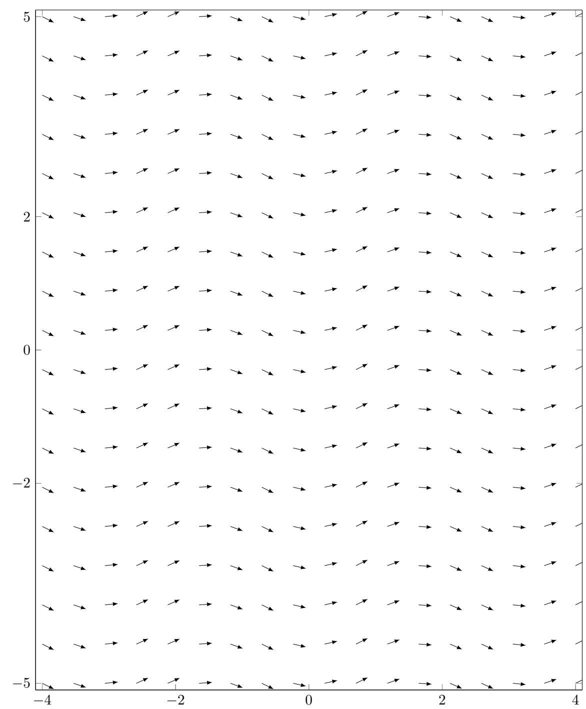
4) $y' = 1$



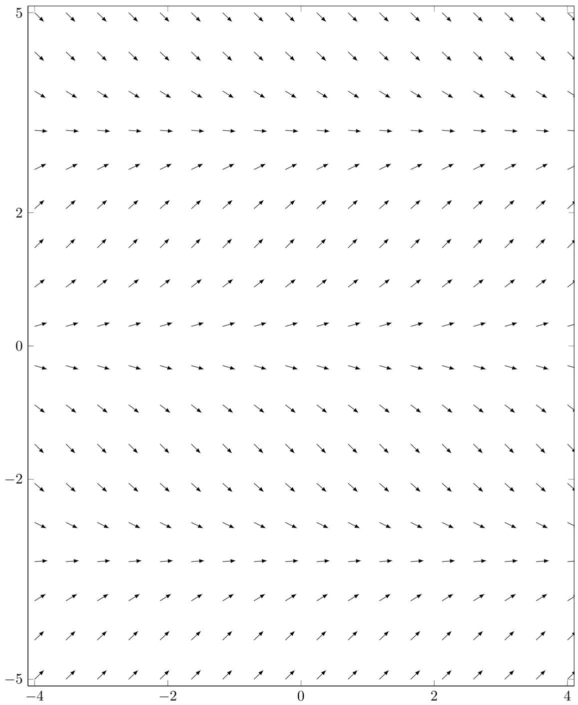
5) $y' = x^2 - y^2$



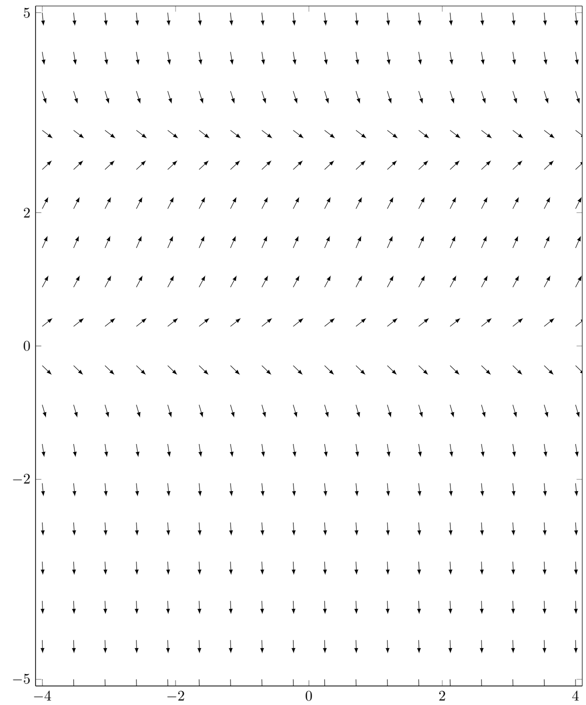
6) $y' = \sin x \cos x$



7) $y' = \sin y$



8) $y' = y(3-y)$



Explanation. It is easy to see that Equations $y' = -2$ and $y' = 1$ match Figures 2) and 4) respectively. Indeed, the general solution of $y' = -2$ is $-2x + C$ and every vector of the associated field has the same "right-down" direction. Similarly, every vector associated with Equation $y' = 1$ has the same "right-up" direction.

The right hand-side in Equations $y' = y(3-y)$ and $y' = \sin y$ is independent of x. This means that vectors on the same horizontal line have the same direction. Only Figures 7) and 8) have such property. (We already excluded Figures 2) and 4).) By analyzing where $y' > 0$ (right-up vectors) and where $y' < 0$ (right-down vectors), we obtain that Equation $y' = y(3-y)$ matches Figure 8) while Equation $y' = \sin y$ matches Figure 7).

The right hand-side in Equations $y' = \sin x \cos x = \dfrac{1}{2}\sin(2x)$ and $y' = |x|$ is independent of y. This means that vectors on the same vertical line have the same direction. Only Figures 3) and 6) have such property. Equation $y' = \sin x \cos x = \dfrac{1}{2}\sin(2x)$ matches Figure 6) as it has a clear periodic pattern.

The remaining equations are $y' = x/y$, $y' = x^2 - y^2$ and $y' = 2x - y$. By analyzing where $y' > 0$ (right-up vectors) and where $y' < 0$ (right-down vectors), we see that $y' = x^2 - y^2$ matches Figure 5) and $y' = 2x - y$ matches Figure 1). □

Problem 12. Find the general solutions to the following second order differential equations

a) $$y'' - 4y' + 4y = 0$$

b) $$y'' - 13y' + 42y = 0$$

c) $$y'' + 9y = 0$$

Solution. **a)** The quadratic equation is $\lambda^2 - 4\lambda + 4 = 0$. It has a unique real root $\lambda = 2$. The general solution:

$$y = C_1 e^{2x} + C_2 x e^{2x}.$$

b) The quadratic equation is $\lambda^2 - 13\lambda + 42 = 0$. It has two real roots $\lambda = 6$ and $\lambda = 7$. The general solution:

$$y = C_1 e^{6x} + C_2 e^{7x}.$$

b) The quadratic equation is $\lambda^2 + 9 = 0$. It has two complex roots $\pm 3i$. The general solution:

$$C_1 \cos(3x) + C_2 \sin(3x)$$

□