Final Exam Practice Problems

MAT 132

Dec 10, 2018

This is a collection of practice problems for final exam in MAT 132. It is much longer than the actual final. Answer key is given at the end.

1. Evaluate the following integrals:

(a)
$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$$

(b)
$$\int \cos^2 x \sin^3 x \, dx$$

(c)
$$\int \frac{x+1}{1+x^2} dx$$

(d)
$$\int \frac{x-8}{x^2-x-6} dx$$

2. Determine whether the following improper integrals are divergent or convergent. You are not required to compute the integral if it is convergent.

(a)
$$\int_{-1}^{1} \frac{dx}{3x+1}$$

(b)
$$\int_{1}^{\infty} \frac{dx}{3x+1}$$

(c)
$$\int_{-\infty}^{\infty} \frac{dx}{x^2+3}$$

(d)
$$\int_{10}^{\infty} \frac{dx}{x\ln(x)}$$

(e)
$$\int_{0}^{\pi} \frac{\sin(x)}{x} dx$$

3. Find the area of the region which lies above x-axis, to the right of $x = 1$ and under the graph of $y = \dfrac{1}{2x^2+5x+2}$.

4. The finite region bounded by $y = x^2$, $x = 0$, and $y = 1$ is rotated about the y-axis. Find the volume of solid of revolution in two different ways: by slicing and by cylindrical shells.

5. The region is situated below the curve $y = e^{-x}$, above the x-axis, and to the right of y-axis. Find the volume of the solid of revolution if the region is rotated a) about the x-axis and b) about the y-axis.

6. Find the length of the curve $9y^2 = 4(x+3)^3$, $y \geq 0$, $0 \leq x \leq 5$.

7. Solve the following differential equations:
(a)
$$xy' = (1 - 4x^2)\tan y$$
(b)
$$xyy' = (y - 1)$$

8. Consider the equation $y' + 2xy - y = 0$.
(a) Find all equilibrium solutions.
(b) Find the general solution of the equation.
(c) Draw several solution curves.
(d) Describe the behavior of solutions as $x \to \infty$. Does it depend on the initial vlaue $y(0)$?

9. Use Eulers method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial value problem
$$y' = xy/2, \qquad y(0) = 1.$$
Find the exact solution $y(x)$ of the initial value problem. Compare the exact value $y(1)$ and its estimation by Eulers method

10. Suppose you take a cold drink out of your refrigerator and set it out in a 70 degree room. When you first take it out, it is 40 degrees. A half hour later it is 50 degrees.

How long after you take it out of the refrigerator will it take for the drink to reach a temperature of 65 degrees?

[Remember that Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference between the temperature of the object and the room it is in.]

11. A tank contains 100 liters of water. Initially all this water is ice cold, at temperature $T_0 = 0$C. At time $t = 0$, we start draining the tank at the rate of 1 L/sec and at the same time, adding hot water at temperature $T_1 = 60$C to the tank, also at the rate of 1 L/sec. The contents of the tank are kept thoroughly mixed.

Determine the tempterature of the water in the tank after 5 minutes.

12. Bismuth-210 is a radioactive material with half-life of 5.0 days. Suppose we start with a sample of bismuth-210 of mass $M(0) = 800$ mg.
(a) Find a formula for the mass remaining after t days.
(b) Find the mass remaining after 30 days.
(c) When is the mass reduced to 1 mg.

13. Determine whether the following sequences converge or diverge. If the sequence converges, find its limit.
(a)
$$a_n = \frac{n^3}{3n^4 - 17n + 2}$$
(b)
$$a_n = \frac{2^{3n}}{3^{2n+1}}$$

(c)
$$a_n = \frac{(-1)^n n^3 + 2n}{n(n+1)(n+2)}$$
(d)
$$a_n = \frac{\sin(n\pi/8)}{2+\sqrt{n}}$$
(e)
$$a_n = n\sin(\pi/n)$$

14. Determine whether the following series converge or diverge. You do not have to find the value of the series if it converges.

(a)
$$\sum_{n=0}^{\infty} \frac{2^{2n} + (-1)^n}{5^n}$$
(b)
$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 2n - 7}$$
(c)
$$\sum_{n=1}^{\infty} n^{-1/3}$$
(d)
$$\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n}$$
(e)
$$\sum_{n=0}^{\infty} \sin(1.2\pi n)$$

15. Find the radius and interval of convergence for the following power series. Note: you are required to determine the behavior at the endpoints of the interval of convergence!

(a)
$$\sum_{n=0}^{\infty} \frac{(3x-1)^n}{4^n}$$
(b)
$$\sum_{n=0}^{\infty} \frac{1+2^n}{n!} x^n$$

16. Write the following functions as power series and determine the radius of convergnce of these series. You are not required to determine the behavior at endpoints of the interval of convergence.

(a) $\dfrac{1}{3x+2}$ (in power series in x)

(b) $\dfrac{1}{3x+2}$ (in power series in $x+1$)

(c) $\dfrac{x}{x^2+9}$ (in power series in x)

(d) $\ln(3x+2)$ (in power series in x)

(e) $\dfrac{1-\cos(x)}{x^2}$

17. Use power series to compute the following integral. Write the answer as a numerical series.

$$\int_0^1 x\cos(x^3)\,dx$$

Answer key

1. (a) $-2\cos(\sqrt{x}) + C$

(b) $\dfrac{\cos^5(x)}{5} - \dfrac{\cos^3(x)}{3} + C$

(c) $\frac{1}{2}\ln(1+x^2) + \tan^{-1}(x) + C$

(d) $2\ln|x+2| - \ln|x-3| + C$

2. (a) divergent

(b) divergent

(c) convergent

(d) divergent

(e) convergent

3. $\ln(2)/3$

4. $\pi/2$

5. a) $\pi/2$; b) 2π

6. $38/3$

7. (a) $y = \sin^{-1}\left(Cxe^{(-2x^2)}\right)$

(b) $y + \ln|y-1| - \ln|x| = C$ or $y = 1$

8. (a) $y = 0$

(b) $y = Ce^{x-x^2}$

(d) For any initial value $y(0)$, we have $\lim_{x\to\infty} y(x) = 0$.

9. Euler's method gives $y(1) \approx 1.214$. Exact solution gives $y(1) = e^{0.25} \approx 1.284$

10. $t = 30\dfrac{\ln(6)}{\ln(3/2)}$ minutes $\approx$ 2 hours 13 minutes

11. $60(1-e^{-3}) \approx 57$ degrees C.

12. (a) $M(t) = 800e^{-t\ln(2)/5} = 800 \cdot 2^{-t/5}$ mg

(b) $M(30) = 800 \cdot 2^{-6} = 12.5$ mg

(c) $t = 5\dfrac{\ln(800)}{\ln(2)} \approx 48.2$ days

13. (a) $\lim a_n = 0$

(b) $\lim a_n = 0$

(c) diverges

(d) $\lim a_n = 0$

(e) $\lim a_n = \pi$

14. (a) converges by ratio test

(b) converges by comparison with $\sum \frac{1}{n^2}$

(c) diverges (p-series)

(d) converges (by ratio test or by comparison with $\sum \frac{1}{3^n}$)

(e) diverges ($\lim a_n$ doesn't exist)

15. (a) $R = 4/3$; interval of convergence is $|x - 1/3| < 4/3$, or $(-1, 5/3)$
(b) $R = \infty$; converges for all values of x

16. (a) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{2^{n+1}} x^n$; $R = 2/3$
(b) $-\sum_{n=0}^{\infty} 3^n (x+1)^n$; $R = 1/3$
(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} x^{2n+1}$; $R = 3$
(d) $\ln(2) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{2^n n} x^n$; $R = 2/3$
(e) $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n+2)!}$; $R = \infty$

17. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!(6n+2)}$