# Definitions of quasiconformality and exceptional sets

Dimitrios Ntalampekos

Stony Brook University

AMS Spring Eastern Sectional Meeting
Special Session on Quasiconformal analysis and geometry on metric spaces

April 2, 2023

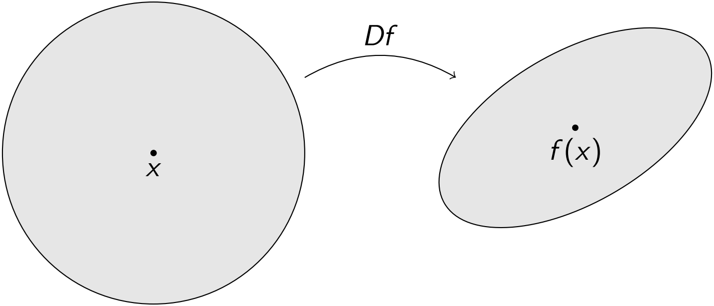
# Classical definitions of quasiconformality

$\Omega \subset \mathbb{R}^n$ open set

$f\colon \Omega \to \mathbb{R}^n$ orientation-preserving topological embedding

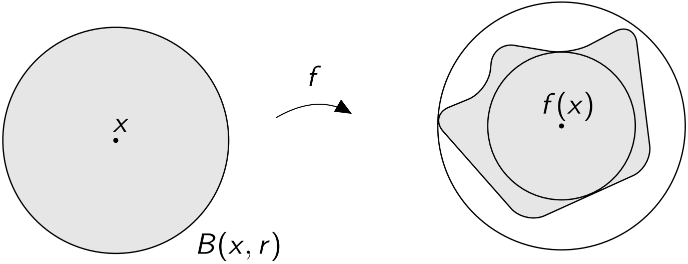
**Definition (Analytic definition)**

$f$ is $K$-quasiconformal if $f \in W^{1,n}_{\text{loc}}(\Omega)$ and $\|Df\|^n \leq KJ_f$ a.e. in $\Omega$.



# Metric definition

$$L_f(x,r) = \sup\{|f(x)-f(y)| : |x-y| \le r\}$$
$$l_f(x,r) = \inf\{|f(x)-f(y)| : |x-y| \ge r\}$$
$$H_f(x) = \limsup_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)}$$




**Theorem (Gehring 1960)**

*f is quasiconformal* $\iff$ $H_f(x) \le H$ *for* **every** $x \in \Omega$.

# Liminf metric definition

$$h_f(x) = \liminf_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)}$$

**Theorem (Heinonen–Koskela 1995)**

*$f$ is quasiconformal $\iff h_f(x) \leq H$ for every $x \in \Omega$.*

We only need to check quasiconformality at a **sequence** of scales!

Applications in rigidity problems in complex dynamics:

Przytycki–Rohde, Graczyk–Smirnov, Haïssinsky, Kozlovski-Shen-van Stiren, Smania

**Question**

*When is a topological conjugacy between dynamical systems (quasi)conformal?*

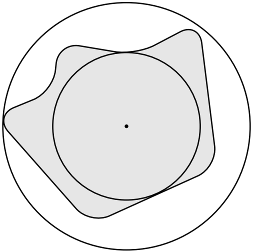
# Generalized metric definition

**Question**

*What if we require that sets of bounded eccentricity are mapped to sets of bounded eccentricity?*

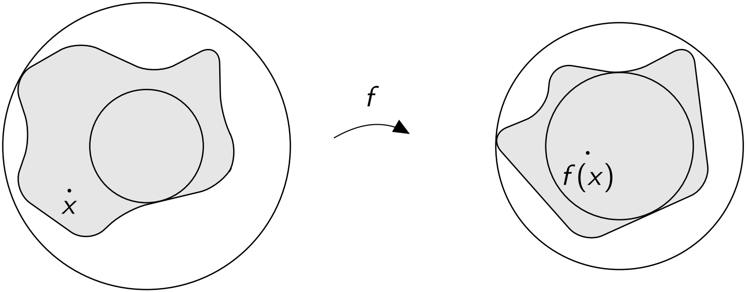
$A \subset \mathbb{R}^n$ bounded open set

$E(A) = \inf\{M \geq 1 : \text{there exists a ball } B \text{ such that } B \subset A \subset MB\}$



# Generalized metric definition

$$E_f(x) = \inf\{M \geq 1 : \text{there exists open sets } A_n \ni x \text{ shirnking to } x, \\ E(A_n) \leq M,\ E(f(A_n)) \leq M\}$$




$E_f \leq h_f \leq H_f$

**Theorem (N. 2021)**

*$f$ is quasiconformal $\iff$ $E_f(x) \leq H$ for every $x \in \Omega$.*

Equivalently: there exist of sets $A_n$, $f(A_n)$ of uniformly bounded eccentricity with $A_n \to x$ and $f(A_n) \to f(x)$.

# Exceptional/removable sets

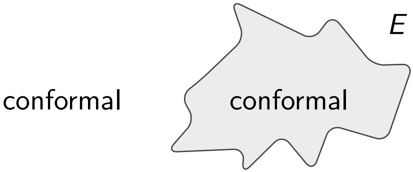
**Question**

*Do we need to assume that $H_f(x), h_f(x)$, or $E_f(x)$ is uniformly bounded at all $x$?*

We cannot remove a set of measure zero! **Too large!**

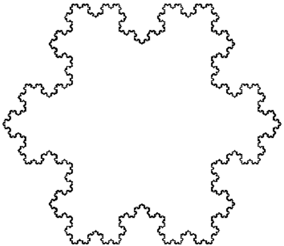
**Definition**

A **closed** set $E$ is **(quasi)conformally removable** if every homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that is (quasi)conformal outside $E$ is (quasi)conformal.



# Removable sets

- Sets of $\sigma$-finite $(n-1)$-measure ($n = 2$: Besicovitch '31; $n > 2$: Gehring '60, Kallunki–Koskela '00, N. '21)
- Quasicircles
- Boundaries of John/Hölder domains (Jones '91, Jones–Smirnov '00, N. '21)
- *NED* sets (Negligible for Extremal Distance) (Ahlfors–Beurling '50)



# NED sets

$\Gamma$ family of paths in $\mathbb{R}^n$
$\rho\colon \mathbb{R}^n \to [0,\infty]$ Borel function
$\rho$ is admissible for $\Gamma$ if $\int_\gamma \rho\, ds \geq 1$ for all rectifiable $\gamma \in \Gamma$.

$$\operatorname{Mod}_n \Gamma = \inf_\rho \int \rho^n$$

## Definition

A set $E \subset \mathbb{R}^n$ is *NED* if

$$\operatorname{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) = \operatorname{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_0(E))$$

for every pair of disjoint continua $F_1, F_2 \subset \mathbb{R}^n$.

$\mathcal{F}_0(E)$= curves in $\mathbb{R}^n$ avoiding $E$ except at the endpoints

# CNED sets

$\mathcal{F}_\sigma(E)$= curves in $\mathbb{R}^n$ intersecting $E$ at countably many points

*CNED*= Countably Negligible for Extremal Distances

**Definition**

A set $E \subset \mathbb{R}^n$ is *CNED* if

$$\mathrm{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) = \mathrm{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_\sigma(E))$$

for every pair of disjoint continua $F_1, F_2 \subset \mathbb{R}^n$.

**Theorem (N. 2021)**

*Suppose that $E \in CNED$ and $E_f(x) \leq H$ for $x \notin E$. Then $f$ is quasiconformal.*

# Examples of CNED sets

All previous examples:

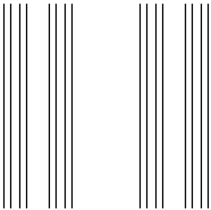
- Rectifiability: Sets of $\sigma$-finite $(n-1)$-measure
- Geometry: Quasicircles, boundaries of John/Hölder domains
- Potential theory: *NED* sets

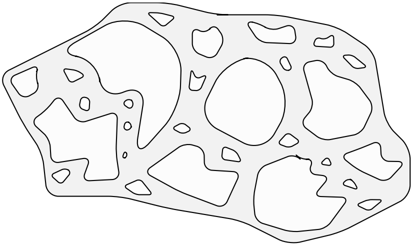
New examples:

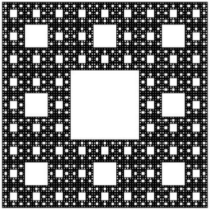
- Non-measurable sets can be *CNED* (Sierpiński 1920)
- Unions of closed *CNED* sets

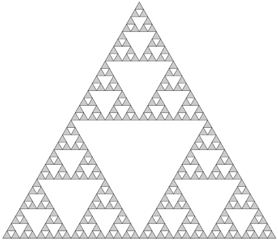
# Examples of non-*CNED* sets

- Sets of positive area
- $C \times [0,1]$
- Sierpiński carpets (also non-removable N. 2019)
- Sierpiński gasket (also non-removable N. 2019)









# Unions of CNED sets

**Question**

*Is the union of two removable closed sets removable?*

Yes for disjoint sets (trivial)

Yes for Cantor sets and quasicircles (Younsi 2016)

**Question**

*Is the union of two CNED sets CNED?*

**Theorem (N. 2021)**

*Suppose that $E_i$ is* **closed** *and $E_i \in (C)NED$ for each $i \in \mathbb{N}$. Then*

$$\bigcup_{i \in \mathbb{N}} E_i \in (C)NED.$$

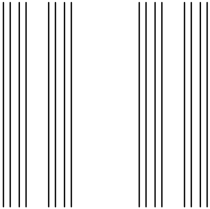
The union can be dense in $\mathbb{R}^n$!

# Unions of CNED sets

**Theorem (N. 2021)**

*There exist* **Borel** *sets* $E_1, E_2 \in NED$ *such that* $E_1 \cup E_2 \notin CNED$.

In fact, $E_1 \cup E_2 = C \times [0,1]$.



$E_1$ = countable union of *NED* Cantor sets $\Rightarrow$ *NED*

$E_2$ = Borel set whose projections to axes have measure zero $\Rightarrow$ *NED*

# Open problems

**Problem**

*Removable sets coincide with CNED sets.*

Implications:

- Union of two removable closed sets is removable.
- Removability is a local condition.
- Removable closed sets coincide with removable sets for continuous $W^{1,2}$ functions.

Thank you!