Carleson measures for the Dirichlet space on the polydisc

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Dirichlet space $\mathcal{D}(\mathbb{D})$

We consider spaces of analytic functions in the unit disc

$$f(z) = \sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} \hat{f}(n) z^n$$

with the norm

$$\|f\|_\alpha^2 = \sum_{n\geq 0} |\hat{f}|^2(n)(n+1)^a, \quad a \in \mathbb{R}.$$

For $a = 0$ we get the Hardy space, and $a = 1$ corresponds to the Dirichlet space,

$$\|f\|_{\mathcal{D}(\mathbb{D})}^2 = \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) + \int_{\mathbb{T}} |f(e^{it})|^2 \, \frac{dt}{2\pi},$$

where $A(\cdot)$ is the normalized surface measure on $\mathbb{D}$. Yet another way to look at the Dirichlet space is to consider analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that the area (counting multiplicities) of $f(\mathbb{D})$ is finite.

Carleson measures

Let H be a Hilbert space of analytic functions on the domain Ω. A measure μ on $\bar{\Omega}$ is called a Carleson measure, if the imbedding $H \mapsto L^2(\bar{\Omega}, d\mu)$ is bounded,

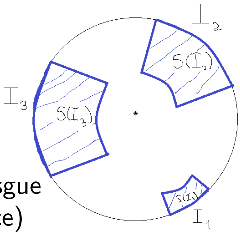
$$\|f\|^2_{L^2(\bar{\Omega}, d\mu)} \lesssim \|f\|^2_H.$$

Theorem (A general one-dimensional 'theorem')

Let $f \in H_a(\mathbb{D})$, where $\|f\|^2_{H_a} = \sum_{n\geq 0} |\hat{f}|^2(n)(n+1)^a$. Then
μ is Carleson for H_a if and only if

$$\mu\left(\bigcup S(I_j)\right) \lesssim \kappa_a\left(\bigcup I_j\right),$$

where $\{I_j\}$ is a finite collection of disjoint intervals on $\mathbb{T}$.



For $a = 0$ (i.e. for H^2) κ_a is the Lebesgue measure, and for $a = 1$ (Dirichlet space) κ is the logarithmic capacity.

Another description

Theorem (Local charge/energy)

Assume that $\operatorname{supp}\mu \subset \mathbb{T}$ *(otherwise we just push it to the boundary). Then* μ *is Carleson for the Dirichlet space on* $\mathbb{D}$ *iff for any dyadic interval* $I \subset \mathbb{T}$ *one has*

$$\sum_{J \subset I} (\mu(J))^2 \lesssim \mu(I).$$

Dirichlet space $\mathcal{D}(\mathbb{D}^2)$

As before, we consider analytic functions on the bidisc $f(z,w) = \sum_{m,n\geq 0} a_{m,n} z^m w^n$. The (unweighted) Dirichlet space on $\mathbb{D}^2$ consists of analytic functions f satisfying

$$\|f\|^2_{\mathcal{D}(\mathbb{D}^2)} = \sum_{m,n\geq 0} (m+1)(n+1)|a_{m,n}|^2 < +\infty.$$

An equivalent definition is

$$\|f\|^2_{\mathcal{D}(\mathbb{D}^2)} = \int_{\mathbb{D}^2} |\partial_{zw} f(z,w)|^2 \, dA(z)\, dA(w) + \int_{\mathbb{D}} \int_{\mathbb{T}} |\partial_z f(z, e^{i\theta})|^2 \, dA(z) \, \frac{d\theta}{2\pi} + \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_w f(e^{it}, w)|^2 \, \frac{dt}{2\pi} \, dA(w) + \int_{\mathbb{T}^2} |f(e^{it}, e^{i\theta})|^2 \, \frac{dt}{2\pi} \, \frac{d\theta}{2\pi}.$$

Suggestion for the general two-dimensional theorem

Let $f \in H_{a,b}(\mathbb{D}^2)$, where $\|f\|^2_{H_{a,b}} = \sum_{m,n\geq 0} |\hat{f}|^2(m,n)(m+1)^a(n+1)^b$.
Then μ is Carleson for $H_{a,b}$ if and only if

$$\mu\left(\bigcup_{k=1}^{N} S(I_k)\times S(J_k)\right) \leq C_\mu \kappa_{a,b}\left(\bigcup_{k=1}^{N} I_k \times J_k\right),$$

where $\{I_k\}, \{J_k\}$ are finite collections of disjoint intervals on $\mathbb{T}$.
As before, for $a=b=0$ (i.e. for $H^2(\mathbb{D}^2)$) $\kappa_{a,b}$ is the Lebesgue measure, and for $a=b=1$ (Dirichlet space) $\kappa_{a,b}$ is the bi-logarithmic capacity.

Local charge/energy for the bidisc

Theorem

Assume that $\operatorname{supp}\mu \subset \mathbb{T}^2$ *(again there is an argument that allows us to do so). Then* μ *is Carleson for the Dirichlet space on* $\mathbb{D}^2$ *iff for any finite collection of dyadic rectangles* $I_k \times J_k \subset \mathbb{T}^2,\ E = \bigcup_{k=1}^N I_k \times J_k$ *one has*

$$\sum_{R \subset E} (\mu(R))^2 \lesssim \mu(E).$$

A plan of sorts

- *Candidate: subcapacitary measures*
- *Preliminary work: duality trick.*
  - We start with boundedness of the imbedding
  - Modification: remove the derivative through RKHS properties
  - Modification: remove the analytic structure
- *Discretize the problem – replace a polydisc $\mathbb{D}^d$ by a "polytree" T^d*
- *Discrete Setting.*
  - Develop appropriate potential theory on T^d
  - Maz'ya approach: reduce the problem to a potential-theoretic statement
  - Reduce the potential-theoretic statement to a combinatorial one
- *Solve the discrete problem and move it back to the polydisc*
- Some possibly related problems.

Potential theory: basics

- Let X, Y be measure spaces, and let $K : Y \times X \to \mathbb{R}$ be a kernel function (subject to some basic conditions). We define

$$V^{\mu}(x) := \int_Y K(y, x)\, d\mu(y).$$

- Newton and Riesz potentials

$$U^{\mu}(x) = \int_{\mathbb{R}^3} \frac{d\mu(y)}{|x - y|}$$
$$I_{\alpha}^{\mu}(x) = \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x - y|^{N-\alpha}}.$$

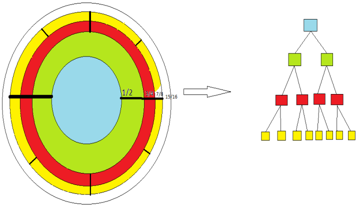
A discrete model of the bidisc

There is a standard way to discretize the unit disc via the Carleson boxes. A resulting discrete object is a uniform dyadic tree T.

The same approach for the bidisc $\mathbb{D} \times \mathbb{D}$ produces the bitree $T \times T$.

A convenient way to represent the dyadic tree T is to consider the system Δ of dyadic subintervals of the unit interval $I_0 = [0, 1)$.

Respectively, the bitree corresponds to the system Δ^2 of dyadic rectangles in $Q_0 = [0, 1)^2$ (and the order relation is again given by inclusion).



Potential theory on the bitree: bilogarithmic potential

We consider measures concentrated on the distinguished boundary $(\partial T)^2$ (no loss of generality here), and all the graphs are finite (say of depth N). Then $(\partial T)^2$ can be identified as a collection of squares $[j2^{-N},(j+1)2^{-N})\times[k2^{-N},(k+1)2^{-N})$.
Let μ be a non-negative Borel measure on $(\partial T)^2$. We define the (bilogarithmic) potential of μ to be

$$V^\mu(\alpha) := \int_{(\partial T)^2} K(\alpha,\omega)\,d\mu(\omega), \quad \alpha\in\bar{T}^2,$$

where $K(\alpha,\omega)=\sharp\{\gamma\in\bar{T}^2:\gamma\geq\alpha,\gamma\geq\omega\}$.
Rectangular representation:

$$V^\mu(Q) = \int_{[0,1)^2} K(Q,x)\,d\mu(x),$$

where Q is a dyadic rectangle, K is as above, and μ has a piecewise constant density on 2^{-N}-sized squares.

Potential theory on the bitree: capacity

In particular, if $y = y(Q)$ is a centerpoint of Q, then

$$K(y, x) \sim \log \frac{1}{|y_1 - x_1|} \log \frac{1}{|y_2 - x_2|},$$

if x and y are "far" enough from each other.
Now let E be a compact subset of the unit square $Q_0 = [0, 1)^2$, we define

$$\operatorname{Cap} E := \inf\{\mathcal{E}[\mu] : V^\mu(x) \geq 1, x \in E\},$$

where

$$\mathcal{E}[\mu] = \int V^\mu \, d\mu$$

is the energy of μ. By the general theory there exists a unique minimizer μ_E — the equilibrium measure of the set E, such that $\operatorname{Cap} E = \mathcal{E}[\mu_E]$ and $V^{\mu_E} \equiv 1$ on $\operatorname{supp} \mu_E \subset E$ (we consider finite bitrees, so no need to deal with q.a.e.).

Potential theory on the bitree: capacitary strong inequality

Now let $\mu \geq 0$, for $\lambda > 0$ consider

$$E_\lambda := \{x \in Q_0 : \ V^\mu(x) \geq \lambda\}.$$

It follows that

$$\mathrm{Cap}\, E_\lambda \leq \mathcal{E}\left[\frac{\mu}{\lambda}\right] = \frac{1}{\lambda^2}\mathcal{E}[\mu],$$

since $\frac{\mu}{\lambda}$ is admissible for E_λ.
Is it true that

$$\int_0^\infty \lambda\, \mathrm{Cap}\, E_\lambda \, d\lambda \leq C\mathcal{E}[\mu],$$

for some absolute constant C?
Maximum Principle:

$$sup_{x\in \mathrm{supp}\,\mu} V^\mu(x) \gtrsim \sup_{x\in Q_0} V^\mu(x),$$

then YES (Maz'ya, Adams, Hansson).

Potential theory on the bitree: capacitary strong inequality

PROBLEM: there exists $\mu \geq 0$ on T^2:

$$1 = \sup_{x \in \operatorname{supp} \mu} V^{\mu}(x) < \sup_{x \in Q_0} V^{\mu}(x) = \infty.$$

SOLUTION (Quantitative MP): if $\operatorname{supp}_{x \in \operatorname{supp} \mu} V^{\mu} \leq 1$ and $\lambda \geq 1$, then

$$\operatorname{Cap} E_{\lambda} \lesssim \frac{1}{\lambda^2 \cdot \lambda} \mathcal{E}[\mu].$$

Equivalent mixed energy estimate: let $F \subset E$, then

$$\mathcal{E}[\mu_E, \mu_F] = \int V^{\mu_E} \, d\mu_F \lesssim (\mathcal{E}[\mu_E])^{\frac{1}{2} - \frac{1}{6}} (\mathcal{E}[\mu_F])^{\frac{1}{2} + \frac{1}{6}}.$$

Further questions

- Possible extensions: $1 \leq p < \infty$, weighted spaces.
- Explore the connections to the multiparameter martingales.
- Related problem — is there a Bellman function technique for the bitree?
- An example. Assume that μ is a probability measure on Q_0. Given $f \in L^2(Q_0, d\mu)$ and $Q \in \Delta^2$ let $\langle f \rangle_Q = \frac{1}{\mu(Q)} \int_Q f\, d\mu$. Define $Mf(x) = \sup_{Q \ni x} \langle f \rangle_Q$ to be the dyadic maximal function. We are interested in the inequality

$$\int_{Q_0} |Mf|^2 \, d\mu \leq C \int_{Q_0} |f|^2 \, d\mu,$$

what conditions one could impose on μ for this inequality to hold?