Project 2

(due Monday, May 13, 2002)

Discuss the questions below in concise and precise essay form. If you use a reference, quote it and do not copy. Use your own words. Problems 2 and 3 go together, but can be done independently. Work on Item 4 is optional and independent, but strongly recommended and will earn extra credit.

1. Redo Problem 3 from Project 1 in detail. A complete proof must contain a uniform continuity argument.

2. For $p \in \mathbf{R}^2$, consider the punctured plane $U_p = \{q \in \mathbf{R}^2 \mid q \neq p\}$. Let c_0, c_1: $[\alpha, \beta] \longrightarrow U_p$ be continuous curves which are both closed at $q_0 = c_0(\alpha) = c_0(\beta) = c_1(\alpha) = c_1(\beta)$. Recall that c_0 and c_1 are called *homotopic* (= deformable into each other) in U_p if there is a continuous mapping (*homotopy* = deformation) H: $[\alpha, \beta] \times [0,1] \longrightarrow U_p$ such that all intermediate curves c_s: $[\alpha, \beta] \longrightarrow U_p$ with $c_s(t) = H(t,s)$ are all closed at $q_0 = c_s(\alpha) = c_s(\beta)$, $0 \leq s \leq 1$.

Show that c_0 and c_1 are homotopic in U_p if and only if we have $W(c_0; p) = W(c_1; p)$ for the corresponding winding numbers. The invariance of winding numbers under deformations says that homotopic curves have equal winding numbers. It remains to prove the converse. Proceed as follows:

(i) Argue that one can assume $p = 0$.

(ii) Argue that after radial projection, one can assume that $|c_0| = |c_1| = 1$ are motions along the unit circle.

(iii) Consider the angle functions φ_0, φ_1 of c_0, c_1 with respect to $p = 0$ so that $\varphi_0(\alpha) = \varphi_1(\alpha) = 0$. Let $\varphi_s = (1-s)\varphi_0 + s\varphi_1$ for $0 \leq s \leq 1$. Why does $H(t,s) = (\cos\varphi_s(t), \sin\varphi_s(t))$ define a homotopy as desired?

3. Consider two regular C^1-curves c_0, c_1: $[\alpha, \beta] \longrightarrow \mathbf{R}^2$ which are (smoothly) C^1-closed. Recall that c_0 and c_1 are called *regularly* homotopic or *isotopic* if there is a *regular* homotopy between them, i.e. a C^1-map H: $[\alpha, \beta] \times [0,1] \longrightarrow \mathbf{R}^2$ such that all deformed curves c_s are regular C^1-closed for all $0 \leq s \leq 1$. The closing point may depend on s. Not that isotopic means intuitively that c_0 can be deformed into c_1 without going through "kinks" or corners, other than self-intersections, which are often present and unavoidable.

Let T_0, T_1 be the unit tangent fields of c_0, c_1. If c_0, c_1 are isotopic, then we had observed already that the continuous closed curves T_0 and T_1 are homotopic as closed curves, and therefore we have for the rotation index $R_0 = W(T_0; 0)$ of c_0 and $R_1 = W(T_1; 0)$ of c_1 that necessarily $R_0 = R_1$.

The converse is true as well. Suppose that $R_0 = R_1$. Then c_0 and c_1 are isotopic (*Graustein-Whitney*). Prove this theorem as follows:

(i) Argue that one can assume c_0 and c_1 are both unit speed curves with the same length $\beta - \alpha$, and furthermore, $T_0(\alpha) = T_1(\alpha) = (1,0)$.

(ii) By 2(iii) above there is a continuous deformation T: $[\alpha, \beta] \times [0,1] \longrightarrow \mathbf{R}^2$ from T_0 into T_1 through continuous curves T_s which arer closed at $q_0 = T_z(\alpha) = T_s(\beta)$ for all $0 \leq s \leq 1$, and T_s depends smoothly on s. Consider

$$\tilde{c}_s(t) = \int_\alpha^t T_s(\tau) d\tau \quad .$$

Why does $(t,s) \longmapsto \tilde{c}_s(t)$ define a regular C^1-homotopy of unit speed curves, which however, are not necessarily closed for $0 < s < 1$?

(iii) In general, a C^1-curve h: $[\alpha, \beta] \longrightarrow \mathbf{R}^2$ is C^1-closed if and only if the velocity field h' is closed and for its vector average we have

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta h'(\tau) d\tau = 0 \ .$$

Now make all fields T_s in (ii) velocity fields of C^1-closed curves by subtracting their vector averages $\overline{T}_s$, where

$$\overline{T}_s = \frac{1}{\beta - \alpha} \int_\alpha^\beta T_s(\tau) d\tau \ .$$

So define

$$\hat{T}_s(t) = T_s(t) - \overline{T}_s \ .$$

(iv) This construction could have created another problem. Clearly, $\hat{T}_s(t)$ will no longer be a unit field. But we have to make sure it vanishes nowhere - in order to construct our regular homotopy. But this follows from the following fact, which you should prove first: If g: $[\alpha, \beta] \longrightarrow \mathbf{R}^2$ is any continuous curve with $|g(t)| = 1$ for all t, i.e. a motion along the unit circle, then the vector average $\overline{g}$ lies in the unit disk, $|\overline{g}| \leq 1$, with equality holding only if g is constant. Thus $g(t) - \overline{g}$ will vanish nowhere if g is non-constant.
Hint: Consider the real vector space E of all continuous functions u: $[\alpha, \beta] \longrightarrow \mathbf{R}$ with the inner pruduct

$$< u, v >= \frac{1}{\beta - \alpha} \int_\alpha^\beta u(\tau) v(\tau) d\tau \ .$$

Denoting by $\|u\| =< u, u >^{1/2}$ the global norm of u, we have the Cauchy-Schwarz inequality

$$< u, v >^2 \leq \|u\|^2 \|v\|^2 \ ,$$

with equality holding iff u, v are linearly dependent. Apply this to a function u and the constant function 1 to obtain

$$< u, 1 >^2 < \|u\|^2 \ ,$$

for u non-constant. How does this imply the claim?

(v) Finally we can define our regular homotopy H by

$$H(t, s) = c_s(t) = \int_\alpha^t \hat{T}_s(\tau) d\tau \ .$$

Carefully explain why this works.

4. Consult the literature and describe a continuous curve c: $[0, 1] \longrightarrow \mathbf{R}^2$ whose image is the whole square $[0, 1] \times [0, 1]$ (*Peano curve*). Why can such a curve never be injective (one-to-one)? Why can it not be C^1?