Solutions

(1)
$$\left[\begin{array}{cccc|c} 1 & 6 & 2 & -5 & 1 \\ -1 & -6 & -1 & -3 & 1 \\ 2 & 12 & 5 & -18 & 4 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 6 & 2 & -5 & 1 \\ 0 & 0 & 1 & -8 & 2 \\ 0 & 0 & 1 & -8 & 2 \end{array}\right]$$

$R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$$\boxed{\begin{aligned} x_1 &= 1 + 5t - 2(2+8t) - 6s \\ &= -3 - 11t - 6s \\ x_2 &= s \\ x_3 &= 2 + 8t \\ x_4 &= t \end{aligned}}$$

(2) $\det A = 4$

$$A^{-1} = \begin{bmatrix} 3/4 & -1 & 9/4 \\ -1/4 & 1 & -7/4 \\ -1/2 & 1 & -3/2 \end{bmatrix}$$

(3)
$$\begin{aligned} -1a + 3b + 2c + d &= 0 \\ a - 2b + 2c + d &= 0 \\ 2a - b + c + d &= 0 \end{aligned}$$

(see end of sols.)

$$\left[\begin{array}{cccc|c} -1 & 3 & 2 & 1 & 0 \\ 1 & -2 & 2 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} -1 & 3 & 2 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 5 & 5 & 3 & 0 \end{array}\right]$$

$R_1 + R_2 \to R_2$
$2R_1 + R_3 \to R_3$

$$\to \left[\begin{array}{cccc|c} 1 & -3 & -2 & -1 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & -15 & -7 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -3 & -2 & -1 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 1 & \frac{7}{15} & 0 \end{array}\right]$$

$(-1)R_1 \to R_1$
$R_3 - 5R_2 \to R_3$

$\left(\frac{1}{-15}\right)R_3 \to R_3$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 10 & 5 & 0 \\ 0 & 1 & 0 & \frac{2}{15} & 0 \\ 0 & 0 & 1 & \frac{7}{15} & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{5}{15} & 0 \\ 0 & 1 & 0 & \frac{2}{15} & 0 \\ 0 & 0 & 1 & \frac{7}{15} & 0 \end{array}\right] \to$$

$R_1 + 3R_2 \to R_1$
$R_2 - 4R_3 \to R_2$

$R_1 - 10R_3 \to R_1$

$$\begin{aligned} a &= -\tfrac{5}{15}t \\ b &= -\tfrac{2}{15}t \\ c &= -\tfrac{7}{15}t \\ d &= t \end{aligned}$$

set $t =$

$$\boxed{\begin{aligned} a &= -5 \\ b &= -2 \\ c &= -7 \\ d &= 15 \end{aligned}}$$

$$\boxed{-5x - 2y - 7z + 15 = 0}$$

(4) a)
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = T_1 \qquad T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T_2 T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

b) $\det T_2 T_1 = -1$

$$(T_2 T_1)^{-1} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

5.

$$3(x^2 - x) + (-2)(2x+1) = 3x^2 - 3x - 4x - 2$$
$$= 3x^2 - 7x - 2$$

<u>dependent</u>

6.

$$\begin{bmatrix} -1 & 2 & 3 & -2 \\ 1 & 2 & -1 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 4 & 2 & 0 \\ 0 & 4 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$(-1)R_1 \to R_1$
$R_1 + R_2 \to R_2$
$R_1 + R_3 \to R_3$

$(1/4)R_2 \to R_2$
$R_3 - R_2 \to R_3$

<u>independent</u>

7.

$$(a+c)\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + (b)\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3a + b + 3c \\ -a + 4b - c \\ 2a + b + 2c \end{bmatrix}$$

8.

$$\begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 1 & -2 & 3 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 4 & 1 \\ 0 & 0 & 4 & 7 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 4 & 1 \\ 0 & 0 & 1 & 7/4 & 5/4 \end{bmatrix}$$

$R_2 \to R_1$
$R_1 + R_2 \to R_2$

$\frac{R_2}{4} \to R_2$

basis for <u>null space</u>:
$$\boxed{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} +5/4 \\ 0 \\ -7/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 11/4 \\ 0 \\ -5/4 \\ 0 \\ 1 \end{bmatrix}}$$

<u>column space</u>:
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \to \boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

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9. a) because they are <u>dependent</u>: $(-2)\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix}$

(10) a) Show: $\det\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \neq 0$

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 0 + 1 = +1$$

b) $3\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}}$

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(11) $\det(\lambda I - A) = 0 \Rightarrow \lambda^3 - 2\lambda = 0$

eigenvalues: $0, -\sqrt{2}, \sqrt{2}$

eigenvectors: $\begin{bmatrix} 5/3 \\ 1/3 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{5(-4+3\sqrt{2})}{5\sqrt{2}-6} \\ \frac{2-\sqrt{2}}{5\sqrt{2}-6} \\ 1 \end{bmatrix},$

$$\begin{bmatrix} \frac{5(4+3\sqrt{2})}{5\sqrt{2}+6} \\ \frac{-2-\sqrt{2}}{5\sqrt{2}+6} \\ 1 \end{bmatrix}$$

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3) $\overrightarrow{P_1P_2} = (2, -5, 0)$
$\overrightarrow{P_1P_3} = (3, -4, -1)$

$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (5, 2, 7)$

$5(x+1) + 2(y-3) + 7(z-2) = 0 \rightarrow \boxed{5x + 2y + 7z - 15 = 0}$