ROCHLIN'S THEOREM, A PROBLEM AND A CONJECTURE

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Closed oriented two manifolds were understood in Riemann's time. Poincaré discovered three manifolds were complicated. Dimensions four, five and more were even more so.

Therefore, it came as a surprise in the 50's that closed manifolds up to cobounding a manifold of one higher dimension could be completely understood in terms of numerical invariants called Pontryagin numbers (integers) and Stiefel-Whitney numbers (integers modulo two).

Rochlin began the pattern by showing in dimension four the cobordism classes of oriented closed smooth manifolds form an infinite cyclic group. The integer, called the signature, attached to M^4 was computed from the intersection of 2-cycles in M^4 as the difference between the number of positive squares and the number of negative squares of the symmetric intersection form. Rochlin proved the formula " the signature equals one-third the first Pontryagin number"

Thom extended this Rochlin pattern to all dimensions using the geometric techniques of **Pontryagin** and Rochlin plus the algebraic topology techniques of **Serre**, showing up to two torsion the class of a manifold was determined by the set of Pontryagin numbers, these being the evaluation of products of Pontryagin classes on the fundamental homology class of the oriented manifold.

Hirzebruch using Thom extended Rochlin's formula in a rich but explicit fashion to all dimensions , for example in dimension 8 the signature is one 45th of (seven times the second Pontryagin number minus the evaluation of the first pontryagin class squared on the fundamental class of the manifold).

Milnor used the seven in that formula to show the 7-sphere had at least seven different smooth structures. The final answer is 28 where the factor of 4 is related to the Dirac operator continuation of Rochlin's contribution discussed below. The figure shows one construction of Milnor's generating exotic seven sphere, which is done by taking the boundary of the eight manifold obtained by connecting up like party rings , tangent disk bundles of the 4-sphere as in the E_8 **Dynkin** diagram.

Back to dimension four.

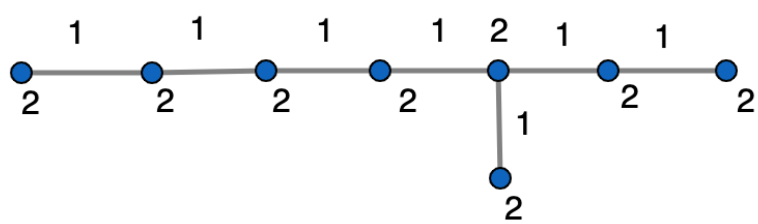
Rochlin's cobordism pattern depended on showing first that the cobordism group in dimension four was determined by the value of the first Pontryagin class evaluated on the fundamental class of the manifold. Then secondly showing the signature of any bounding manifold had to be zero. This last proposition is elementary but one of the most important facts in manifold topology.

But the most profound point comes now.

Rochlin also calculated by a geometric argument a la Pontryagin that if M^4 was almost parallelizable, ie parallelizable in the complement of a point, then the first Pontryagin number was actually divisible by 48. Thus the signature of such a closed four manifold , which Rochlin proved was one third of the first Pontryagin number, had to be divisibly by 16. This divisibility by 16 is the celebrated **Rochlin Theorem** about smooth four manifolds.

This was at first glance a curious result for the following reason: being almost parallelizable for the oriented closed four manifold meant exactly that the self intersection number of any 2-cycle was even, the value mod two being determined by evaluating the second Stiefel-Whitney class on the cycle.

The intersection form was non-degenerate over the integers by Poincaré duality. Such even unimodular forms inside all quadratic forms taking integral values were studied in number theory. There it was known these properties meant the signature was divisible by 8 and by no more in general. A basic example being the E_8 matrix where the (inner) products for a special basis is illustrated by the E_8 Dynkin diagram:



Where each nodal basis element has self intersection number 2 and two nodal basis elements intersect exactly once if and only if there is an edge between them , otherwise the inner product is zero. E_8 is an even unimodular symmetric form of signature 8.

One knows that E_8 generates the indefinite even unimodular forms in the sense any such form is a direct sum of E_8's and hyperbolic forms $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Thus Rochlin's theorem shows half of the elements in the infinite set of even indefinite unimodular forms cannot appear as the intersection form of any smooth closed four manifold. Namely those with an odd number of E_8's. An example that does appear is the ubiquitous K3 complex surface whose intersection form is two E_8 's and three hyperbolic forms. .

This result set the stage for another important development in Topology, Geometry and Analysis.

This relates to definite forms.

In number theory one also knows that there are finitely many unimodular definite symmetric forms of a given rank, the number growing exponentially with the rank.

Donaldson proved none of those definite forms except the identity form occurs as the intersection form of a smooth four manifold. This is the first theorem of the unexpected Donaldson theory discovered three decades after Rochlin's theorem.

Freedman at the same time showed remarkably that every unimodular form occurs for closed topological four manifolds.

Donaldson theory does not prove Rochlin's theorem because Rochlin's statement involves hyperbolic forms.

In fact there is an intermediate class of manifolds between smooth and topological where the analysis of Donaldson theory is perfectly valid.

There are two such intermediate classes of manifolds , the ones with coordinate charts where the transition mappings are bi-Lipschitz, and ones where the transition mappings are quasiconformal.

Lets call these **Sobolev** manifolds.

Problem:
Is Rochlin's theorem true for these Sobolev four manifolds?

Conjecture:
If Rochlin's theorem is true for Sobolev four manifolds , then Sobolev four manifolds are actually smoothable.

Information:
Closed topological four manifolds are almost smoothable , namely they are smoothable in the complement of a point (see surveys and book by **Frank Quinn.**)
Also outside dimension four all topological manifolds carry unique Sobolov structures of each type.
The proof makes heavy use of the **Kirby-Edwards** completely elementary and very ingenious construction of paths of homeomorphisms between nearby homeomorphisms in all dimensions (late 60's).
These paths allowed **Seibenmann** '69 to construct higher dimensional manifold counterexamples to the Hauptvermutung soon after he understood the precise role played by Rochlin's Theorem about four dimensions in this question.

Operators on Hilbert Space
The signature operator twisted by a vector bundle exists in the Sobolev context. The unbounded version exists in the Lipschitz context. The bounded version, just using the phase of the operator (which contains all of the topological information), exists in the quasiconformal context. Stiefel- Whitney classes make sense in these settings so the possibility of constructing Dirac operators also makes sense. This is unknown at present (more below).

Physics
Donaldson theory is part of a larger quantum field theory which has an effective version obtained by integrating out certain variables.
This effective version has expression in terms of Dirac operators which depend on the tangent bundle. One knows Rochlin's theorem can be deduced in a context using Dirac operators, the Atiyah-Singer Index theorem and quaternions (more below).
Physicists believe Donaldson theory and its effective version **Seiberg-Witten** theory are equivalent. From the perspective of Sobolev manifolds Rochlin's theorem provides a challenge to and an opportunity for understanding better this belief.

More history:

In the middle 60's this author as a second year Princeton topology grad student was following the evidently powerful constructive cobordism techniques of **Browder-Novikov** classifying smooth manifolds in a **homotopy type with stable tangent vector bundle specified** plus the covering space method of **Novikov** for showing the rational Pontryagin classes were homeomorphism invariants. The motivation was to study firstly , **PL manifolds in a given homotopy type** without PL stable tangent micro bundle specified and secondly to study **PL manifolds in a given homeomorphism type** without PL stable tangent microbundle specified. These formulations suggested by the influence of Milnor and **Steenrod** had completely calculable outcomes, whereas every other formulation did not have such completely calculable outcomes.

Given a homotopy equivalence $f : L \to M$ one could define in all dimensions numerical obstructions to f being homotopic to a PL-homeomorphism via differences of signatures of V and $f^{-1}V$ where V is a manifold cycle in M and f^{-1} is its transversal preimage in L. These differences were divisible by 8 because f is a homotopy equivalence and so pulls back Stiefel Whitney classes. There were also modulo n versions of this picture where V is a mod n manifold cycle.

The vanishing for a finite generating set of these characteristic invariants of f was necessary for f to be homotopic to a homeomorphism , and further to be homotopic to a PL homeomorphism if when for the mod n characteristic cycles of dimension four the division by 8 was upgraded to a division by 16 using Rochlin's Theorem. In higher dimensions than four this vanishing and this refined vanishing were also respectively sufficient in the simply connected case.

(This description for simplicitly has absorbed the mod two Arf-Kervaire invariants in dim 4k-2, (first encountered for k=1 by Pontryagin in his misstep of '42) into the mod two signature invariants in dimension 4k by crossing them with RP^2, described in the work with **John Morgan**, Annals of Math,1972)

The refined vanishing sufficiency was achieved in 66' for the PL homeomorphism case ((''On the Hauptvermutung for Manifolds ''BAMS July '67) and the vanishing sufficiency became valid for the homeomorphism case as a corollary in '69 of the general topological manifold theory achieved by Kirby-Siebenmann.

The Rochlin refinement gave an order two class in the integral fourth cohomology of L canonically defined when f is a homeomorphism. This seems an appropriate time to name this heretofore unnamed class **the order two integral Rochlin class** in the four dimensional cohomology with integer coefficients.

In the hands of **Kirby-Seibenmann** the entire difference between the PL and topological manifold categories in higher dimensions could be completely understood by the profound factor of 2 implied by Rochlin's 16. They proved '69 the homeomorphism f was connected by a path of homeomorphisms to a PL-homemorphism (higher dimensions and no simply connected hypothesis required) iffi a **"mod two Rochlin class"** in the degree three cohomology of L with Z/2Z coefficients vanished, and all of these classes, referred to as Kirby-Siebenmann classes are realized by geometric examples.

These two Rochlin classes, the mod two Rochlin type class in degree three obstructing an isotopy of the homeomorphism to a Pl homeomorphism and the integral Rochlin class of order two in degree four obstructing a homotopy of the homeomorphism to a PL homeomorphism are related by the integral Bockstein operation. The Bockstein operation takes an integral cochain representative of the mod two class and forms 1/2 of its coboundary to obtain an integral cocycle in degree four (so that two times it is obviously a coboundary).

This "Bockstein of the mod two Kirby-Siebenmann class is the order two integral Rochlin class" discussion is related to the important discovery in recent times by **Manolescu** reported at this conference of the existence of higher dimensional topological manifolds not homeomorphic to a triangulated topological manifold.

More information for the Rochlin problem and the Rochlin conjecture:

Work of Kirby-Edwards (mentioned above) and work of Kirby depending on that of Novikov was used to show '76 that topological manifolds in all dimensions, except for dimension four, could be provided with unique Sobolev structures of either type. This used a substitution of the d-torus used in those works by an almost parallizable closed hyperbolic d-manifold. ("Hyperbolic Geometry and Homeomorphisms" in the book "Geometric Topology" 1979 Academic Press).

Interestingly, the existence of these almost parallelizable hyperbolic manifolds depends on an argument learned from **(Deligne and Mazur)** that the algebraic topology modulo n of a complex algebraic variety can be defined for the algebraic variety reduced mod p for p prime and not dividing n and not involved awkwardly in the defining equations of the variety.

After the opposite results of Donaldson and Freedman in '82 it was natural to ask about their results for the intermediate class of Sobolev four manifolds. The answer was: Donaldson theory works for both classes of Sobolev four manifolds. ("Quasiconformal 4-Manifolds" Acta Mathematica 1989).

In studying Rochlin's Theorem in the Sobolev context , it is useful to know that the index theorem holds there (**N.Teleman**) and that there are local representatives for the Pontryagin classes defined using the bounded phase of the signature operator in **Alain Conne's** perspective of non commutative geometry. ("Quasiconformal mappings, Operators on Hilbert Space and Local formulae for Characteristic Classes" Topology 1994).

Considerations related to the construction of Dirac operators and the context of smooth versus Sobolev manifolds plus a smoothability and a Dirac operator conjecture are discussed in "Foundations of Geometry, Analysis and the Differentiable Structure for Manifolds" in the book "Low Dimensional Topology" World Scientific 1999.