

# Quantitative towers in finite difference calculus approximating the continuum

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*In memory of Sir Michael Atiyah*

**Abstract** Multivector fields and differential forms at the continuum level have respectively two commutative associative products, a third composition product between them and various operators like  $\partial$ ,  $d$  and “star” which are used to describe many nonlinear problems. The point of this paper is to construct consistent direct and inverse systems of finite dimensional approximations to these structures and to calculate combinatorially how these finite dimensional models differ from their continuum idealizations. In a Euclidean background there is an explicit answer which is natural **statistically**.

## 1 Introduction

At the continuum level there is a rich nonlinear structure (see Background Appendix) with more symmetry than is possible for any system of finite dimensional approximations.

The discrete vector calculus discussed here is based on combinatorial topology with an unexpected Grassmann algebra fit. One will have at each scale two finite dimensional graded commutative associative algebras: one product on chains and one product on cochains. There is a third contraction product between chains and cochains. The chains have their differential  $\partial$  of degree  $-1$  and the cochains have their differential  $\delta$  of degree  $+1$ . Neither  $\partial$  nor  $\delta$  is a derivation of its product. Also  $\partial$  is further away by one scale order of magnitude from being a derivation than  $\delta$  is.

In periodic Euclidean-space the new point [8] is to consider  $2^n$  overlapping scale  $2h$ -cubical grids where  $h = 2^{-i-1}$  at level  $i$  to build finite-dimensional spaces of combinatorial chains and cochains. This particular choice allows discrete vector and covector Grassmann products and their contraction product. The boundary operator  $\partial$  on chains and the coboundary operator  $\delta$  on cochains are precisely related by the Poincaré duality “star operator”.

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The operators do not “derive” their product structures as they do in the continuum analogue, but their higher order deviations from such, called “infinitesimal cumulants”, have a noteworthy  $h$ -divisibility structure developed here.

Namely, one finds for each level of discretization  $i$ , the finite-dimensional algebras indicated above, after extending scalars by power series in the formal scale parameter  $h$ , with their operators  $\delta$  for cochains and  $\partial$  for chains, satisfy  $h$  divisibility properties that make them respectively into consistent inverse and direct systems of binary QFT algebras. See §§6,7 below. Also similar estimates control the **traditional cumulants** of the linear mappings respecting boundary operators between the commutative algebras at different scales.

We will apply to each finite dimensional algebra with differential, a general procedure in order to resolve the broken symmetry of continuum algebra.

The procedure involves higher-order brackets which are Taylor coefficients of coderivations. These are described in §2 and [1]. The details are new for the  $\partial$  case and for the  $\delta$  case in that the structure resonates with Jae-Suk Park’s algebraic background and computational prescription (2018) for perturbative QFT [2] referred to as binary QFT algebras. These are described in terms of divisibility of the brackets above by appropriate powers of the scale variable. This binary QFT formulation at the finite rank algebra level extends the theoretical physicist’s Batalin-Vilkovisky formalism that requires the continuum.

**Definition 1.1.** *If a graded commutative associative algebra over the formal power series in a formal variable  $h$  has a differential, this determines a structure consisting of all the brackets referred to above (which turn out to be expressible in terms of commutators of the differential with iterated products; see example 2.5). If these all vanish this is called a classical differential algebra. If the  $(k + 1)$ -bracket is **divisible** by  $h^k$  for all  $k \in \mathbb{N}$ , this is called a **binary QFT algebra**.*

We put ourselves in the context of this definition by extending the scalars of the chains and cochains in the construction by the formal power series in  $h$ . We add relations to this free module making the differences divisible by  $h$ , these differences being defined using the shifts by  $h$  in the coordinate directions of the lattice. More details are in §§6,7. The proofs fill §§2,3,4,5,6,7.

**Theorem 1.2.**

- A) *The sequence of bracket operators of  $\partial$  on lattice chains referred to above and described in §2 are divisible by the following powers of  $h$ :  $1, 1, 2, 3, \dots$ . This means by Definition 1.1, that the operator  $\partial$  on the chains over formal power series in  $h$  with the difference quotients added, defines a binary QFT algebra.*
- B) *There are natural linear mappings from a coarse scale to a finer scale respecting the  $\partial$  operators and there are canonical structure-preserving mappings between the infinitesimal cumulant bracket structures defined at each*

scale. The Taylor components of these coalgebra mappings are the usual cumulants of statistics and are divisible by appropriate powers of  $h$ .

**Theorem 1.3.**

- A) The sequence of bracket operators of  $\delta$  on lattice cochains have scale orders  $1, 2, 3, \dots$  in powers of the scale  $h$ . This means by Definition 1.1 that the operator  $h^{-1}\delta$  on the cochains with scalars extended to the formal power series in  $h$  with difference quotients added, defines a binary QFT algebra.
- B) There are natural linear mappings from a fine scale to a coarser scale respecting the  $\delta$  operators and there are canonical structure-preserving mappings between the infinitesimal cumulant bracket structures defined at each scale. The Taylor components of these coalgebra mappings are the usual cumulants of statistics and are divisible by appropriate powers of  $h$ .

**Takeaway:** The calculations below will prove these theorems and show that as the grid size tends to zero, the infinitesimal cumulants of the  $h^{-1}\partial$  structure on chains with respect to the product, tends to zero starting at the 3-bracket. The  $h^{-1}\partial$  structure converges to the  $\partial$  differential geometric algebra discussed in the Background Appendix, where the commutator with  $\partial$  there includes the Lie bracket of vector fields.

For cochains the quadratic term also tends to zero, with the  $h^{-1}\delta$  operator becoming approximately a derivation of the exterior product on cochains, and the entire structure approaches the differential algebra of differential forms, also with the features as discussed in the Background Appendix.

Finally, these algebras of chains and cochains make up consistent direct and inverse systems with explicit structure-preserving mappings. The Taylor coefficients of the mappings between scales are the cumulants of random variables in statistics and these satisfy appropriate divisibility by powers of  $h$ .

The entire package in a Euclidean background might be thought of as a universal explicit prescription for treating the interesting closure problem in finite dimensional approximations to nonlinear problems. It might substitute for the “effective action” approximations of theoretical physics when the action principle is not apparent or not paramount.

**The layout of the paper is as follows.** §2 recalls for the reader the multi-bracket structure defined by the cumulant bijection and builds most of the combinatorics needed for the proofs. The geometry of the lattice beginning in three-dimensions is explained in §3. The divisibility by powers of  $h$  computations are carried out on the lattice for  $(\wedge, \delta)$  in §4 and for  $(\wedge, \partial)$  in §5. Chain and cochain mappings are given in §7 between the algebras of chains on lattices at two different scales and extended to give **binary QFT algebra** morphisms, i.e. with the estimates, between the higher bracket structures at various scales.

**Acknowledgements** The structure of the cellular combinatorics from [8] and computations of its cumulant structure here, yield the  $h$ -divisibility properties. These in turn require the special nature (§2.2, §2.3) of the cumulant bijection, discussed in [5] and in [4]. The functorial aspects of the cumulant bijection are used first in [7] and then in §7 to obtain a direct and an inverse system of binary QFT algebras in each dimension  $n$  and with morphisms between scales.

The cumulant bijection was directly inspired by a CUNY Einstein Chair online lecture of Jae Suk Park in 2012 which eventually become his formulation of **binary QFT algebras** in 2018 (see [2] pp 19,20). The results here are for specific combinatorial constructions from which the continuum algebra emerges. Some aspects fit with Markl [1] in the context of homotopical algebra. The constructions here are canonical, explicit and rigid. We would like to thank the referee for their comments on making the exposition more focused and succinct.

## 2 General constructions

Suppose that  $V$  is a graded commutative associative algebra  $(V, m)$  with square-zero map  $\partial : V \rightarrow V$  of degree  $\pm 1$ . The  $\mathbb{Z}$ -grading in  $V$  is denoted by  $|\cdot|$ . Write  $\pm^{x,y}$  for the sign  $(-1)^{|x|\cdot|y|}$ . (Graded) commutativity means as usual that  $yx = \pm^{x,y}xy$  where we write  $xy$  for  $m(x \otimes y)$ .

In this section we will discuss the infinitesimal cumulants of  $\partial$  (namely its higher deviations from being a derivation) and in particular see how they can be viewed as “Taylor coefficients of a coderivation” of square zero on  $S^*V$ , with “...” being explained below. To keep the paper self-contained, to establish our notation and to give the formulae for the first few terms we have written subsections 2.1-2.3 with their proofs. Much of this is familiar to various experts. See [1] for an interesting general perspective. The specific estimates here on statistical cumulants using these subsections plus §§3,4,5,7 were the goal and seem to be new.

### 2.1 Comultiplication on $S^*V$

Consider the tensor algebra  $T^*V = \bigoplus_{k=1}^{\infty} V^{\otimes k}$ . The grading on  $V$  induces a grading on  $T^*V$ , so that  $x_1 \otimes \dots \otimes x_k$  has grading  $|x_1| + \dots + |x_k|$ . There is a (graded) coassociative comultiplication  $\Delta$  on  $T^*V$  of degree zero defined by

$$\Delta(x_1 \otimes \dots \otimes x_k) = \sum_{i=1}^{k-1} (x_1 \otimes \dots \otimes x_i) \otimes (x_{i+1} \otimes \dots \otimes x_k)$$

Consider  $S^kV \subset V^{\otimes k}$  as the subspace spanned by

$$x_1 \wedge \dots \wedge x_k \equiv \sum_{\sigma \in S_k} \pm^{(\sigma,x)} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)}$$

for  $x_1, \dots, x_k \in V$ , where  $\pm^{(\sigma, x)} \equiv \prod_{i>j, \sigma^{-1}(u)<\sigma^{-1}(j)} (-1)^{|x_i| \cdot |x_j|}$ . The grading on  $T^*V$  induces a grading on  $S^*V$ , so that  $x_1 \wedge \dots \wedge x_k$  has grading  $|x_1| + \dots + |x_k|$ . The coassociative comultiplication  $\Delta$  defined above on  $T^*V$  induces a cocommutative coassociative comultiplication of degree 0 on  $S^*V$ . Explicitly,

$$\Delta(v_1 \wedge \dots \wedge v_k) = \sum_I \pm^I v_I \otimes v_{I^c}$$

the sum being over all  $2^k - 2$  proper non-empty subsets  $I$  of  $[k] \equiv \{1, \dots, k\}$ , whose complement is denoted  $I^c$ , while if the elements of  $I$  are written in increasing order as  $I = \{i_1, \dots, i_r\}$  then  $v_I \equiv v_{i_1} \wedge \dots \wedge v_{i_r}$ . The sign is

$$\pm^I = \prod_{i \in I, j \in I^c, i > j} (-1)^{|v_i| \cdot |v_j|} \quad (1)$$

Verification of cocommutativity,  $P \circ \Delta = \Delta$  where  $P(v \otimes w) = (-1)^{|v| \cdot |w|} w \otimes v$  follows from the identity  $\pm^I \pm^{I^c} = (-1)^{|v_I| \cdot |v_{I^c}|}$ .

*Example 2.1.* The comultiplication applied to an element of  $S^3V$  is a sum of  $2^3 - 2 = 6$  terms. Thus  $\Delta(v_1 \wedge v_2 \wedge v_3)$  is

$$\begin{aligned} & v_1 \otimes (v_2 \wedge v_3) + (-1)^{|v_1| \cdot |v_2|} v_2 \otimes (v_1 \wedge v_3) + (-1)^{(|v_1|+|v_2|)|v_3|} v_3 \otimes (v_1 \wedge v_2) \\ & + (v_1 \wedge v_2) \otimes v_3 + (-1)^{|v_1| \cdot |v_2|} (v_2 \wedge v_1) \otimes v_3 + (-1)^{(|v_1|+|v_2|)|v_3|} (v_3 \wedge v_1) \otimes v_2 \end{aligned}$$

## 2.2 Cumulant bijection

Let  $\tau_1 : S^*V \rightarrow V$  be the multiplication, that is,  $\tau_1(v_1 \wedge \dots \wedge v_n) = v_1 \cdots v_n$ . Following [5], define the *cumulant bijection*  $\tau : S^*V \rightarrow S^*V$  to be the coalgebra lift of  $\tau_1$  to  $S^*V$ . This means that  $\tau \equiv \sum_{r=1}^{\infty} \tau_r$  where  $\tau_r : S^*V \rightarrow S^rV$  is defined for  $r > 1$  by

$$\tau_r = (\tau_1)^{\otimes r} \circ \Delta^{r-1}$$

where  $\Delta^{r-1} : S^*V \rightarrow (S^*V)^{\otimes r}$  is the iterated coproduct defined by  $\Delta^1 = \Delta$  and  $\Delta^{n+1} = (\Delta \otimes \text{id}^{\otimes n}) \circ \Delta^n$ . That  $\tau : S^*V \rightarrow S^*V$  is a coalgebra map means that  $(\tau \otimes \tau) \circ \Delta = \Delta \circ \tau$ .

*Example 2.2.* On monomials of order up to three, the formulae for  $\tau$  are

$$\begin{aligned} \tau(a) &= a \\ \tau(a \wedge b) &= ab + a \wedge b \\ \tau(a \wedge b \wedge c) &= abc + a \wedge bc + (-1)^{|a| \cdot |b|} b \wedge ac + (-1)^{(|a|+|b|)|c|} c \wedge ab + a \wedge b \wedge c \end{aligned}$$

As a linear map  $\tau : S^*V \rightarrow S^*V$ , its action is block upper-triangular where the  $r$ -th block is  $S^rV$ , and the diagonal blocks are the identity. Thus  $\tau$  is invertible and its inverse looks like

$$\begin{aligned} \tau^{-1}(a) &= a \\ \tau^{-1}(a \wedge b) &= a \wedge b - ab \\ \tau^{-1}(a \wedge b \wedge c) &= a \wedge b \wedge c - a \wedge bc - (-1)^{|a| \cdot |b|} b \wedge ac - (-1)^{(|a|+|b|)|c|} c \wedge ab + 2abc \end{aligned}$$

In particular, the coefficients in the formulae for  $\tau$  are always  $\pm 1$  while those in the formulae for  $\tau^{-1}$  are more complicated, depending on the relevant partition. The component of  $\tau^{-1}(v_1 \wedge \cdots \wedge v_k)$  in  $V$  (with no wedges) is a multiple (depending on  $k$ ) of the product  $\tau_1(v_1 \wedge \cdots \wedge v_k) = v_1 \cdots v_k$ .

**Lemma 2.3.** *On  $S^k V$ , the actions of  $\tau$  and  $\tau^{-1}$  are given explicitly by*

$$\begin{aligned}\tau(v_1 \wedge \cdots \wedge v_k) &= \sum_{r=1}^k \sum_{I_1 \cup \cdots \cup I_r = [k]} \pm \tau_1(v_{I_1}) \wedge \cdots \wedge \tau_1(v_{I_r}) \\ \tau^{-1}(v_1 \wedge \cdots \wedge v_k) &= \sum_{r=1}^k \sum_{I_1 \cup \cdots \cup I_r = [k]} \left( \prod_{p=1}^r d_{|I_p|} \right) \pm \tau_1(v_{I_1}) \wedge \cdots \wedge \tau_1(v_{I_r})\end{aligned}$$

where  $d_k \equiv (-1)^{k-1}(k-1)!$ , the summations are over all (unordered) partitions of  $[k]$  into  $r$  non-empty subsets, and the sign is that acquired in changing the order of the  $v$ 's from sequential in  $v_1 \wedge \cdots \wedge v_k$  to that in  $v_{I_1} \wedge \cdots \wedge v_{I_r}$ . In particular,  $p_1 \circ \tau^{-1} = (-1)^{k-1}(k-1)!\tau_1$ , where  $p_1$  denotes the projection map  $S^*V \rightarrow V$  on the first component.

*Proof.* The formula for  $\tau$  follows immediately from its definition above,  $\tau = \sum_{r=1}^{\infty} ((\tau_1)^{\otimes r} \circ \Delta^{r-1})$  when combined with the action of the coproduct in §2.1. To prove the formula for  $\tau^{-1}$ , it suffices to verify that its composition with  $\tau$  is the identity. Let  $X$  denote the expression given on the right hand side of the second equation in the statement of the lemma; it suffices to prove that  $\tau(X) = v_1 \wedge \cdots \wedge v_k$ . Computing, using the formula for  $\tau$ ,

$$\tau(X) = \sum_{r=1}^k \sum_{I_1 \cup \cdots \cup I_r = [k]} \sum_{s=1}^r \sum_{J_1 \cup \cdots \cup J_s = [r]} \frac{1}{r!s!} \left( \prod_{p=1}^r d_{|I_p|} \right) \pm \tau_1(v_{K_1}) \wedge \cdots \wedge \tau_1(v_{K_s})$$

where  $K_a \equiv \bigcup_{p \in J_a} I_p$ . Here and throughout the rest of this proof, the summations are over ordered partitions and hence extra factors have been introduced (reciprocals of factorials). Interchanging the summations over  $s$  and  $r$  and extracting the summation over  $K_1, \dots, K_s$  leaves

$$\sum_{s=1}^k \sum_{K_1 \cup \cdots \cup K_s = [k]} \pm \frac{1}{s!} \tau_1(v_{K_1}) \wedge \cdots \wedge \tau_1(v_{K_s}) \cdot \sum_{r=s}^k \frac{1}{r!} \sum_{\substack{I_1 \cup \cdots \cup I_r = [k] \\ J_1 \cup \cdots \cup J_s = [r] \\ K_a = \bigcup_{p \in J_a} I_p}} \left( \prod_{p=1}^r d_{|I_p|} \right)$$

the inner summation being over partitions  $\{I_p\}$  and  $\{J_a\}$  which match given  $\{K_j\}$ . It suffices to show that the coefficient on the right hand side vanishes unless  $s = k$  and the  $K_a$  are singletons when it is 1. For a fixed partition  $K_1 \cup \cdots \cup K_s = [k]$  and any (ordered) partition  $\{J_a\}$  of  $[r]$ , a matching partition  $\{I_p\}$  is given by splitting each  $K_a$  ( $a = 1, \dots, s$ ) into  $|J_a|$  subsets. The coefficient

of the term coming from the particular partition  $K_1 \cup \dots \cup K_s$  is thus

$$\sum_{r=s}^k \frac{1}{r!} \sum_{J_1 \cup \dots \cup J_s = [r]} \prod_{a=1}^s \left( \sum_{\substack{\text{ordered partitions of} \\ K_a \text{ into } |J_a| \text{ sets}}} \prod_{p=1}^{|J_a|} d_{|p^{\text{th}} \text{ set}|} \right)$$

The inner sum depends on  $\{J_a\}$  only by the orders of the sets, giving

$$\sum_{r=s}^k \frac{1}{r!} \sum_{j_1 + \dots + j_s = r} \frac{r!}{\prod_{a=1}^s j_a!} \prod_{a=1}^s \left( \sum_{\substack{\text{ordered partitions of} \\ K_a \text{ into } j_a \text{ sets}}} \prod_{p=1}^{j_a} d_{|p^{\text{th}} \text{ set}|} \right)$$

Interchanging the first two summations leaves

$$\sum_{j_1, \dots, j_s \geq 1} \prod_{a=1}^s \left( \frac{1}{j_a!} \sum_{\substack{\text{ordered partitions of} \\ K_a \text{ into } j_a \text{ sets}}} \prod_{p=1}^{j_a} d_{|p^{\text{th}} \text{ set}|} \right)$$

which is a product over  $a = 1, \dots, s$  of

$$\sum_{j \geq 1} \frac{1}{j!} \sum_{\substack{\text{ordered partitions of} \\ K_a \text{ into } j \text{ sets}}} \prod_{p=1}^j d_{|p^{\text{th}} \text{ set}|} = \sum_{j \geq 1} \frac{1}{j!} \sum_{b_1 + \dots + b_j = |K_a|} \frac{|K_a|!}{\prod_{p=1}^j b_p!} \prod_{p=1}^j d_{b_p}.$$

Taking out the constant  $|K_a|!$  from the summations and using the formula  $d_b = (-1)^{b-1}(b-1)!$ , the inner sum is seen to become the coefficient of  $x^{|K_a|}$  in  $(\sum_{b=1}^{\infty} (-1)^{b-1} \frac{1}{b} x^b)^j = (\ln(1+x))^j$  and so the whole expression sums to  $|K_a|!$  times the coefficient of  $x^{|K_a|}$  in  $\sum_{j \geq 1} \frac{1}{j!} (\ln(1+x))^j = x$  which therefore vanishes unless  $|K_a| = 1$  and in that case is 1.  $\square$

### 2.3 Higher infinitesimal cumulants

Extend the differential  $\partial$  on  $V$  to a coderivation on  $S^*V$  with respect to  $\Delta$ . Explicitly, define  $\partial^\wedge : S^*V \rightarrow S^*V$  by,

$$\partial^\wedge(v_1 \wedge \dots \wedge v_k) \equiv \sum_{i=1}^k (-1)^{\sum_{j < i} |v_j|} v_1 \wedge \dots \wedge \partial v_i \wedge \dots \wedge v_k$$

Since  $\partial$  is a square-zero map of degree  $\pm 1$ , the same follows about  $\partial^\wedge$ . Further,  $\partial^\wedge$  is a coderivation on  $S^*V$  with respect to  $\Delta$ , meaning that

$$\Delta \circ \partial^\wedge = (\partial^\wedge \otimes \text{id} + \overline{\text{id}} \otimes \partial^\wedge) \circ \Delta$$

where  $\overline{\text{id}} : S^*V \rightarrow S^*V$ ,  $\overline{\text{id}}(v) = (-1)^{|v|}v$ . Since  $\tau$  preserves the grading, it commutes with  $\overline{\text{id}}$ .

The conjugation of a coderivation by a coalgebra isomorphism is a coderivation, so that  $D \equiv \tau^{-1} \partial^\wedge \tau$  is a square-zero coderivation of  $S^*V$ . Furthermore,

a coderivation  $D$  on  $S^*V$  is uniquely determined by its projection on the first factor  $p_1 \circ D : S^*V \rightarrow V$ . Thus it is determined by its homogeneous components  $\partial_k \equiv (p_1 \circ D)|_{S^kV} : S^kV \rightarrow V$ , known as the **Taylor components or coefficients** of  $D$ .

**Definition 2.4.** [5] Write  $D \equiv \tau^{-1} \partial^\wedge \tau$  and denote its  $k^{\text{th}}$  Taylor component  $S^kV \rightarrow V$  by  $\partial_k$ . We will call  $\partial_k$  the  $k$ -bracket induced by the pair of graded commutative associative algebra  $(V, m)$  and square-zero map  $\partial$ .

The map  $\partial_{k+1}$  will be referred to as the  $k^{\text{th}}$  infinitesimal cumulant of  $\partial$  with respect to the multiplication  $m$ ; this is reasonable because they measure infinitesimally the deviation of the exponential from being a (bijective) product-preserving mapping, as can be seen in the formulae for  $\partial_k$  given in the following example.

*Example 2.5.* On monomials,  $D$  acts by

$$\begin{aligned} D(a) &= \partial a \\ D(a \wedge b) &= \partial(ab) - \partial a \cdot b + \partial a \wedge b + (-1)^{|a| \cdot |b|} (\partial b \wedge a - \partial b \cdot a) \end{aligned}$$

while  $D(a \wedge b \wedge c)$  is a sum of 19 terms. Its Taylor coefficients are

$$\begin{aligned} \partial_1(a) &= \partial a \\ \partial_2(a \wedge b) &= \partial(ab) - \partial a \cdot b - (-1)^{|a| \cdot |b|} \partial b \cdot a \\ \partial_3(a \wedge b \wedge c) &= \partial(abc) - \partial(ab) \cdot c - (-1)^{|b| \cdot |c|} \partial(ac) \cdot b - (-1)^{|a|(|b|+|c|)} \partial(bc) \cdot a \\ &\quad + (-1)^{(|a|+|b|)|c|} \partial c \cdot ab + (-1)^{|a| \cdot |b|} \partial b \cdot ac + \partial a \cdot bc \end{aligned}$$

For example,  $\partial_2$  is the first infinitesimal cumulant and expresses the (first) deviation from Leibniz, equivalently  $\partial_2(a \wedge b) = \partial(ab) - \partial a \cdot b - (-1)^{|a|} a \cdot \partial b$ .

**Lemma 2.6.** [5] The map  $D$  is a square-zero coderivation of  $S^*V$  and its  $k^{\text{th}}$  Taylor coefficient is given by the following formula where  $\pm^I$  is as defined in (1)

$$\partial_k(v_1 \wedge \cdots \wedge v_k) = \sum_{r=1}^k (-1)^{k-r} \sum_{|I|=r} \pm^I \partial(\tau_1(v_I)) \cdot \tau_1(v_{I^c}) \quad (2)$$

Here the inner summation is over order  $r$  subsets  $I \subseteq [k]$ .

*Proof.* By Definition 2.4,  $D = \tau^{-1} \circ \partial^\wedge \circ \tau$  from which it follows immediately that  $D$  has square zero and is a coderivation. Its  $k^{\text{th}}$  Taylor coefficient is  $\partial_k = p_1 \circ D = p_1 \circ \tau^{-1} \circ \partial^\wedge \circ \tau$  on  $S^kV$ , which we compute from Lemma 2.3 for  $\tau$ ,

$$\partial_k(v_1 \wedge \cdots \wedge v_k) = (p_1 \circ \tau^{-1} \circ \partial^\wedge) \sum_{r=1}^k \sum_{I_1 \cup \cdots \cup I_r = [k]} \pm \tau_1(v_{I_1}) \wedge \cdots \wedge \tau_1(v_{I_r})$$

where the sum is over all (unordered) partitions of  $[k]$  into  $r$  non-empty subsets, and the sign is that acquired in changing the order of the  $v$ 's from sequential in



$v_1 \wedge \cdots \wedge v_k$  to that in  $v_{I_1} \wedge \cdots \wedge v_{I_r}$ . Equivalently, introducing a factor  $\frac{1}{r!}$ , one may sum over ordered partitions of  $[k]$  into  $r$  non-empty subsets, which we now do. By the definition of  $\partial^\wedge$  above, this expands as

$$(p_1 \circ \tau^{-1}) \sum_{r=1}^k \sum_{I_1 \cup \cdots \cup I_r = [k]} \sum_{p=1}^r \pm \frac{1}{r!} \partial(\tau_1(v_{I_p})) \wedge \tau_1(v_{I_1}) \wedge \cdots \widehat{\tau_1(I_p)} \wedge \cdots \wedge \tau_1(v_{I_r})$$

where the hat indicates a missing term, and the sign is that acquired by the (new) change in order of the  $v$ 's, where  $v_{I_p}$  is now listed first. Applying Lemma 2.3 for  $\tau^{-1}$ , this becomes

$$\begin{aligned} & \sum_{r=1}^k \sum_{I_1 \cup \cdots \cup I_r = [k]} \sum_{p=1}^r \pm (-1)^{r-1} \frac{1}{r} \cdot \partial(\tau_1(v_{I_p})) \tau_1(v_{I_p^c}) \\ &= \sum_{\substack{I \subseteq [k] \\ I \neq \emptyset}} \sum_{r=1}^k \sum_{p=1}^r \sum_{\substack{I_1 \cup \cdots \cup I_r = [k] \\ I_p = I}} \pm^I (-1)^{r-1} \frac{1}{r} \cdot \partial(\tau_1(v_I)) \tau_1(v_{I^c}) \\ &= \sum_{\substack{I \subseteq [k] \\ I \neq \emptyset}} \sum_{r=1}^{k+1-|I|} (-1)^{r-1} N_{k-|I|, r-1} \cdot \pm^I \partial(\tau_1(v_I)) \tau_1(v_{I^c}) \end{aligned}$$

where  $N_{s,t}$  denotes the number of ordered partitions of a set of order  $s$  into  $t$  non-empty sets, or equivalently the number of onto maps  $[s] \rightarrow [t]$ .

To complete the proof of the lemma, we need to show that the inner sum is  $(-1)^{k-|I|}$ , that is that  $\sum_{r=1}^{s+1} (-1)^{r-1} N_{s, r-1} = (-1)^s$  for  $s \geq 0$ . For  $s = 0$ , this is  $N_{0,0} = 1$  while for  $s > 0$ , it is the identity

$$N_{s,s} - N_{s,s-1} + \cdots \pm N_{s,1} = 1 \quad (3)$$

For completeness, we include a proof. By the inclusion-exclusion principle, we can count onto maps  $[s] \rightarrow [t]$  by counting all maps ( $t^s$ ) and excluding those in the union of  $t$  sets, namely those whose image does not include  $1, 2, \dots, t$ , respectively. This gives

$$N_{s,t} = t^s - t \cdot (t-1)^s + \binom{t}{2} (t-2)^s - \cdots = \sum_{k=1}^t (-1)^{t-k} \binom{t}{k} k^s$$

Taking an alternating sum and interchanging the summations now gives

$$\sum_{t=1}^s (-1)^{s-t} N_{s,t} = \sum_{k=1}^s \sum_{t=k}^s (-1)^{s-k} \binom{t}{k} k^s = \sum_{k=1}^s (-1)^{s-k} \binom{s+1}{k+1} k^s$$

which is the coefficient of  $x^s$  in  $(1-x)^{s+1} \sum_{k=1}^{\infty} k^s x^k = (1-x)^{s+1} \cdot \left(x \frac{d}{dx}\right)^s \left(\frac{1}{1-x}\right)$ .

Denote this expression by  $p_s(x)$  so that  $\left(x \frac{d}{dx}\right)^s \left(\frac{1}{1-x}\right) = \frac{p_s(x)}{(1-x)^{s+1}}$  and observe

that  $x \frac{d}{dx} \left( \frac{p_{s-1}(x)}{(1-x)^s} \right) = \frac{sx \cdot p_{s-1}(x) + x(1-x) \cdot p'_{s-1}(x)}{(1-x)^{s+1}}$  so that  $p_0(x) = 1$  while  $p_s = sx \cdot p_{s-1} + x(1-x)p'_{s-1}$ . We see inductively that  $p_s$  is a polynomial of degree  $s$ . The recurrence relation on  $p_s$  induces one on its leading coefficient  $a_s$ , namely  $a_s = sa_{s-1} - (s-1)a_{s-1} = a_{s-1}$  while  $a_0 = 1$ . Thus  $a_s = 1$  for all  $s$ . This completes the proof of (3).  $\square$

**Lemma 2.7.** *Fixing the last  $(k-2)$  components,  $\partial_k$  as a map  $V \otimes V \rightarrow V$  is the commutator of  $\partial_{k-1}$  (as a map  $V \rightarrow V$ ) with multiplication; that is,*

$$\begin{aligned} \partial_k(v_1 \wedge v_2 \wedge v_3 \wedge \dots \wedge v_k) &= \partial_{k-1}(v_1 v_2 \wedge v_3 \wedge \dots \wedge v_k) \\ &\quad - (-1)^{|v_1|} v_1 \cdot \partial_{k-1}(v_2 \wedge v_3 \wedge \dots \wedge v_k) \\ &\quad - (-1)^{|v_2|(1+|v_1|)} v_2 \cdot \partial_{k-1}(v_1 \wedge v_3 \wedge \dots \wedge v_k) \end{aligned}$$

*Proof.* By Lemma 2.6, the left hand side can be written as a sum of terms indexed by non-empty subsets  $I \subseteq [k]$ . This sum can be split according as 1 and/or 2 are elements of  $I$ ,

$$\partial_k(v_1 \wedge \dots \wedge v_k) = \left( \sum_{\substack{I \subseteq [k], I \neq \emptyset \\ 1, 2 \notin I}} + \sum_{\substack{I \subseteq [k] \\ 1, 2 \in I}} + \sum_{\substack{I \subseteq [k] \\ 1 \in I, 2 \notin I}} + \sum_{\substack{I \subseteq [k] \\ 2 \in I, 1 \notin I}} \right) \pm^I (-1)^{k-|I|} \partial(v_I) \cdot v_{I^c}$$

Here we omitted the symbols  $\tau_1$  denoting repeated multiplication, which is being written as concatenation. Denote these four sums  $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$  respectively. Similarly, each term on the right hand side of the equality to be proved can be expanded using lemma 2.6 as a sum of  $2^{k-1} - 1$  terms, over non-empty subsets of a  $(k-1)$ -element set, and further each of these sums can be split into two according as the subset does not or does contain the first element. This gives the three terms on the RHS of the equality to be proved as,  $-\Sigma_1 + \Sigma_2, \Sigma_1 + \Sigma_4$  and  $\Sigma_1 + \Sigma_3$ , respectively.  $\square$

This lemma will be useful in proving the inductive steps when we come to compute  $\partial_k$  on the lattice in §4 and §5.

**Definition 2.8.** *When the  $k^{\text{th}}$  infinitesimal cumulant of  $\partial$  vanishes (that is,  $\partial_{k+1} = 0$ ) we say that  $\partial$  has Grothendieck order at most  $k$  with respect to  $m$ .*

Extending  $\partial_k : S^k V \rightarrow V$  to a coderivation on  $S^*V$  yields a map  $\partial_k^\wedge : S^*V \rightarrow S^*V$  which more precisely acts as  $\partial_k^\wedge : S^r V \rightarrow S^{r-k+1} V$  and has  $D = \sum_{k=1}^\infty \partial_k^\wedge$ . The general formula for the action of  $\partial_n^\wedge$  is (where  $\pm^I$  is as defined in (1))

$$\partial_n^\wedge(v_1 \wedge \dots \wedge v_k) = \sum_{\substack{I \subseteq [k] \\ |I|=n}} \pm^I \partial_n(v_I) \wedge v_{I^c} \quad (4)$$

The map  $\partial_k^\wedge : S^*V \rightarrow S^*V$  has Grothendieck order at most  $k$  with respect to the multiplication  $\wedge$  on  $S^*V$ .

*Example 2.9.*  $\partial_1^\wedge = \partial^\wedge$ , while  $\partial_2^\wedge$  acts on monomials up to third order by

$$\begin{aligned}\partial_2^\wedge(a) &= 0 \\ \partial_2^\wedge(a \wedge b) &= \partial(ab) - \partial a \cdot b - (-1)^{|a| \cdot |b|} \partial b \cdot a \\ \partial_2^\wedge(a \wedge b \wedge c) &= \partial(ab) \wedge c + (-1)^{|a|} a \wedge \partial(bc) + (-1)^{|b| \cdot |c|} \partial(ac) \wedge b \\ &\quad - \partial a \cdot b \wedge c - (-1)^{|b| \cdot |c|} \partial a \cdot c \wedge b - (-1)^{|a| \cdot |b|} \partial b \cdot a \wedge c \\ &\quad - (-1)^{|a|} a \wedge \partial b \cdot c - (-1)^{(|a|+|b|) \cdot |c|} \partial c \cdot a \wedge b - (-1)^{(|a|+|b|) \cdot |c|} a \wedge \partial c \cdot b \\ &= \partial_2(a \wedge b) \wedge c + (-1)^{|a|} a \wedge \partial_2(b \wedge c) + (-1)^{|b| \cdot |c|} \partial_2(a \wedge c) \wedge b\end{aligned}$$

In conclusion, we have the following general structure to be estimated in §§4,5,7 for the example of §3.

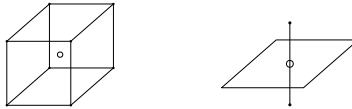
**Theorem 2.10.** *For any graded commutative associative algebra  $(V, m)$  with square-zero map  $\partial$  of grading  $\pm 1$ ,  $S^*V$  is a cocommutative coassociative coalgebra with maps  $\partial_k^\wedge : S^r V \rightarrow S^{r-k+1} V$  as defined above, for which  $D = \sum_{k=1}^\infty \partial_k^\wedge$  is a coderivation of square zero while each  $\partial_k^\wedge$  has Grothendieck order at most  $k$  with respect to  $\wedge$  and  $\partial_1^\wedge = \partial^\wedge$ . The Taylor coefficients of  $D$  are maps  $\partial_k : S^k V \rightarrow V$  which define  $k^{\text{th}}$  order multi-brackets and describe the  $(k-1)^{\text{th}}$  infinitesimal cumulant of  $\partial$  (higher order deviation from Leibniz).*

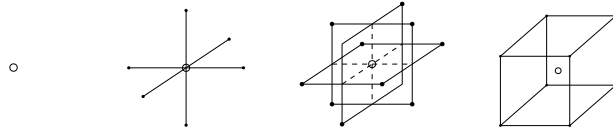
### 3 The geometry of the lattice chain algebra and the lattice cochain algebra

**Discretization of the continuum algebra in dimension three.** In [8], one builds the vector calculus of the lattice model defined by all of the  $2h$  cubes in the  $h$  cubical lattice of  $\mathbb{R}^3$ , from all of these larger cubes, their faces, their edges and their vertices by forming vector spaces  $C_k$  which are “up to sign” generated by these so-called  $k$ -cubes, for  $k = 3, 2, 1$  and  $0$ . For  $k = 0$ ,  $C_k$  is actually generated by all of the  $h$ -lattice points and for  $k = 1, 2, 3$ ,  $C_k$  is generated by **oriented**  $k$ -cubes of edge length  $2h$  modulo the relations  $(k\text{-cube, orientation}) = -(k\text{-cube, opposite orientation})$ . These are the *chains*.

One obtains finite dimensionality by boundary conditions, here by identifying periodically, with period which is the same power of two in each direction.

The dual spaces  $C^k$ , consisting of linear functionals on these finite dimensional  $C_k$ , are called *cochains*. There is also a star operator  $*$  on each, changing  $k$  to  $3 - k$ , and if one identifies chains and cochains using the basis, then star conjugates  $\partial$  to  $\delta$  and *vice versa* since  $*^2 = 1$ . See the upper row of the figure.





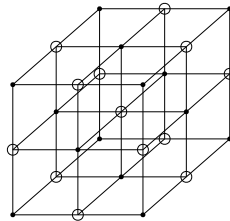
**The main point of the choice of combinatorics.** Firstly, there is for cochains, defined on  $k$ -cubes of side length  $2h$ , a graded commutative and associative product which makes use of the linear algebra of alternating tensors. This is so because at each lattice point there are exactly  $(1, 3, 3, 1)$   $k$ -cubes for  $k = 0, 1, 2, 3$ , of edge length  $2h$  with centre point at that grid point. See the second row of the figure. This gives a direct sum decomposition of the  $2h$ -cochains on the  $h$ -lattice into the exterior algebras of the cotangent spaces of  $\mathbb{R}^3$  at the lattice points. There is then the obvious direct sum exterior algebra structure on cochains, which is graded commutative and associative.

The operator  $\delta$  is not a derivation of the product, which relation only appears on taking the calculus limit. The error is order  $h$ , because there are  $h$ -shifts in the true Leibniz type formula for  $\delta$  of a product.

Secondly, the  $h$ -lattice chains generated by the bigger  $k$ -cubes of edge size  $2h$  have a direct sum decomposition into the exterior algebras on the tangent spaces at the lattice points. Thus the chains also have a graded commutative and associative algebra structure. This product is not familiar in combinatorial or algebraic topology but its continuum analogue appears in differential geometry.

One considers for chains, as one does for continuum multivector fields, a second product or bracket  $[ , ]$ , defined as the deviation of  $\partial$  on chains from being a derivation of the commutative and associative product on chains. Tautologically,  $\partial$  is a derivation of this bracket because  $\partial^2 = 0$ . Note this means the bracket  $[ , ]$  of two cycles is not only a cycle but is canonically a boundary, being  $\partial$  of the exterior product of the two cycles. (See Appendix for the continuum discussion.)

**Discretisation in  $n$ -dimensions.** Consider an  $n$ -dimensional periodic cubic lattice with grid step  $h$  and a size divisible by four in each direction. That is, the vertices are  $L_h = (h\mathbb{Z})^n / (4Nh\mathbb{Z})^n$  which is naturally bicoloured by  $\frac{1}{h} \sum_{i=1}^n x_i$  modulo 2 at  $(x_i) \in L_h$ . This bicolouring splits the lattice as a union of two interlocking lattices, here pictured for  $n = 3$ .



Consider the subcomplex of the chain complex whose  $r$ -chains are generated by  $r$ -dimensional cubes of edge length  $2h$  (whose vertices all have the same colour).

Such an  $r$ -cube is specified by its centre, a point in  $L_h$ , and its type, the subset of  $[n]$  consisting of those directions for which the projection of the cube on the corresponding axis is an interval of length  $2h$  (rather than a point). Since these cubes are Cartesian products of intervals, choosing an ordering on  $[n]$  will induce natural orientations on the cubes.

Let  $C$  denote the graded vector space of chains, graded by dimension, with basis  $\{I_a\}$ , where  $I_a$  labels the cell of type  $I \subset [n]$  whose centre is at lattice point  $a$ . Denote by  $Ia$  the translation of  $a$  forward by  $h$  in each of the directions in  $I$  and by  $I'a$ , the translation of  $a$  backward by  $h$  in each of the directions in  $I$ . By abuse of notation, when  $I$  is a singleton  $\{u\}$ , we will write  $Ia$  and  $I'a$  as  $ua$  and  $u'a$ , respectively. The geometric boundary is given on generators by

$$\partial(I_a) = \sum_{u \in I} \pm_I^u ((I \setminus u)_{ua} - (I \setminus u)_{u'a})$$

where the sign  $\pm_I^u$  is such that  $u \cdot (I \setminus u) = \pm_I^u \cdot I$ , that is  $\pm_I^u = (-1)^{\#\{i \in I \mid i < u\}}$ . The homology of this chain complex has dimension  $2^n$  in degree zero because of the way we have constructed cells of edge length  $2h$ .

**The chain algebra.** The vector space of chains  $C$  naturally splits as a direct sum  $C = \oplus_{a \in L_h} C_a$  where  $C_a = \wedge^* \mathbb{R}^n$  consists of those chains which are combinations of cubical cells centred at  $a$ . The (pointwise) wedge product makes  $C$  into a graded commutative associative algebra.

A generic chain can be written as

$$\sum_{I,a} f_I(a) \cdot I_a$$

where we combine the coefficients  $f_I(a)$  of the basis chains  $I_a$  into scalar-valued functions  $f_I$  on lattice points, defined for all  $2^n$  subsets  $I$  of  $[n]$ .

**The cochain algebra.** Let  $A$  denote the graded vector space of scalar-valued cochains on the lattice (vector space dual to  $C$ ). An element of  $A$  is (determined by) a function  $f$  which evaluates on chains  $I_a$  to a scalar  $f(I_a)$ , or equivalently  $2^n$  functions  $f_I$ , each a scalar-valued function on the lattice (here  $f_I(a) \equiv f(I_a)$ ). One could write the corresponding cochain as

$$\sum_I f_I dI$$

using the notation of differential forms. The graded vector space  $A$  splits naturally as  $A = \oplus_{a \in L_h} A_a$  where each  $A_a = \wedge^* \mathbb{R}^n$  consists of the cochains supported on cells centred at  $a$ . Use the pointwise wedge product to define a multiplication,  $m$ , on  $A$ ; on zero-cochains it gives pointwise multiplication of scalar valued functions on the lattice. Thus if  $f, g \in A$ , their product  $f.g \equiv m(f, g)$  is defined by

$$(f.g)(I_a) = \sum_{J \cup K = I} \pm_{J,K} f(J_a)g(K_a) \quad (5)$$

where the sum is over all ordered pairs of disjoint sets  $J, K$  for which  $J \cup K = I$ . The sign  $\pm_{J,K}$  is defined so that  $J \cdot K = \pm_{J,K}(J \cup K)$ , that is,  $\pm_{J,K} = (-1)^{\#\{(j,k)|j \in J, k \in K, j > k\}}$ . If  $f$  is an  $r$ -cochain and  $g$  is an  $s$ -cochain, then  $f.g$  will be an  $(r+s)$ -cochain while  $g.f = (-1)^{rs}f.g$ . So  $m$  is a (graded) commutative associative product on  $A$ .

Since the cubic lattice is self-dual, there is a natural correspondence between the chain complex and the cochain complex. Thus operators can be interchangeably considered as acting on the chain complex or the cochain complex. In Lemma 3.1, we choose to write all the operators as acting on cochains.

**The star operator  $*$ .** The star operator acts on basis elements by

$$*(dI) = \pm_{I,I^c} dI^c$$

where  $I^c$  denotes the complement  $[n] \setminus I$ , so that  $(dI) \wedge *(dI) = d[n]$ . Since  $\pm_{I,I^c} \cdot \pm_{I^c,I} = (-1)^{|I| \cdot |I^c|}$ , thus  $*$  :  $A^k \rightarrow A^{n-k}$  while  $*^2 = (-1)^{k(n-k)}$  on  $k$ -cochains. This means that for odd  $n$ ,  $*$  is an involution. On arbitrary cochains,

$$(*f)_I = \pm_{I^c,I} f_{I^c} \tag{6}$$

**The map  $\partial$ .** Let  $T_u$  and  $T'_u$  denote translations by  $h\mathbf{e}_u$  and  $-h\mathbf{e}_u$  respectively, where  $\mathbf{e}_u$  is the unit vector in  $\mathbb{R}^n$  in the positive  $u$ -direction. These operators act on the lattice  $L_h$  and thus also on functions on the lattice by

$$(T_u f)(p) \equiv f(p - h\mathbf{e}_u), \quad (T'_u f)(p) \equiv f(p + h\mathbf{e}_u)$$

A general chain can be written as a linear combination  $\sum_{I,a} f_I(a) I_a$ , where  $f_I$  is a scalar-valued function on the lattice  $L_h$ , for each  $I \subseteq [n]$ . The geometric boundary  $\partial$  acts on general chains as

$$\partial \left( \sum_{I,a} f_I(a) I_a \right) = \sum_{I,a} \sum_{u \in I} \pm_u^I f_I(a) ((I \setminus u)_{ua} - (I \setminus u)_{u'a}) = \sum_{I,a} \sum_{u \notin I} [(T_u - T'_u) f_{uI}](a) I_a$$

where we have extended the coefficients  $f_I$  to be defined for arbitrary sequences of elements of  $[n]$ , rather than subsets  $I$ , in such a way that they vanish if the sequence contains a repeat, the sign changes under transposition of adjacent elements and is equal to the original  $f_I$  when the sequence is monotonic increasing;  $uJ$  is the concatenation of  $u$  with  $J$  in that order. Since  $f$  vanishes on sequences containing a repeat, the restriction  $u \notin I$  can be lifted, leaving

$$\partial = \sum_{u=1}^n \partial_u \quad \text{where} \quad \partial_u \left( \sum_I f_I \cdot I \right) = \sum_I (T_u - T'_u) f_{uI} \cdot I$$

Here we have omitted  $a$  from the notation. This can be written concisely as

$$\partial = \sum_{u=1}^n \partial_u \quad \text{where} \quad (\partial_u f)_I = (T_u - T'_u) f_{uI} \tag{7}$$

**The map  $\delta$ .** The coboundary  $\delta : A^r \longrightarrow A^{r+1}$  can be defined by its action on monomials

$$\delta(fdI) = \sum_{u \notin I} (T'_u f - T_u f) du \wedge dI$$

Since  $du \wedge dI = 0$  for  $u \in I$ , the restriction on  $u \notin I$  can be dropped and

$$\delta = \sum_u \delta_u \quad \text{where} \quad \delta_u(fdI) \equiv (T'_u f - T_u f) du \wedge dI = \delta_u(f) \wedge dI \quad (8)$$

where on the right hand side,  $f$  is considered as the 0-cochain  $fd\emptyset$ . On general cochains,

$$(\delta_u f)_I = \begin{cases} \pm_I^u (T'_u - T_u) f_{I \setminus u} & \text{if } u \in I \\ 0 & \text{if } u \notin I \end{cases} \quad (9)$$

**Lemma 3.1.** Define  $\bar{\text{id}} : A \longrightarrow A$  by  $\bar{\text{id}}(fdI) = (-1)^{|I|} fdI$ . With  $m, *, \partial$  and  $\delta$  as defined above in (5), (6), (7) and (9), and  $\bar{*} = * \circ \bar{\text{id}}$ ,

- (i)  $\delta_u \circ \bar{*} = * \circ \partial_u$
- (ii)  $\partial_u \circ * = \bar{*} \circ \delta_u$
- (iii)  $\delta_u \circ m = m \circ (\delta_u \otimes T'_u + (\bar{\text{id}} \circ T_u) \otimes \delta_u)$  (shifted analogue of Leibniz for  $\delta$ )

*Proof.* (i) By direct calculation  $(*(\partial_u f))_I = \pm_{I^c, I} (T_u - T'_u) f_{uI^c}$  while

$$(\delta_u(*f))_I = \begin{cases} \pm_I^u \pm_{(I \setminus u)^c, I \setminus u} (T'_u - T_u) f_{(I \setminus u)^c} & u \in I \\ 0 & u \notin I \end{cases}$$

so that  $(\delta_u(*f))_I = (-1)^{n+1-|I|} (*\partial_u f)_I$ .

(ii) We obtain  $(*(\delta_u f))_I = (-1)^{n-|I|} (\partial_u(*f))_I$  in a similar way to (i).

(iii) It is sufficient to verify on monomials,

$$\begin{aligned} \delta_u((fdI) \cdot (gdJ)) &= (T'_u - T_u)(fg) du \wedge dI \wedge dJ \\ &= ((T'_u f)(T'_u g) - (T_u f)(T_u g)) du \wedge dI \wedge dJ \\ &= (T'_u f - T_u f) du \wedge dI \cdot T'_u g dJ + (-1)^{|I|} (T_u f) dI \cdot (T'_u g - T_u g) du \wedge dJ \\ &= \delta_u(fdI) \cdot T'_u(gdJ) + (-1)^{|I|} T_u(fdI) \cdot \delta_u(gdJ) \quad \square \end{aligned}$$

*Remark 3.2.* There is also a shifted analogue of Leibniz for  $\delta$  (Lemma 3.1 (iii)) where the shifts are in the opposite directions, that is with  $T_u$  and  $T'_u$  interchanged.

*Remark 3.3.* Observe that in the continuum limit  $h \longrightarrow 0$  with  $Nh$  constant, the geometry just discussed reduces to that of the torus  $\mathbb{T}^n$  while the operators  $*, \frac{1}{2h}\delta$  and  $\frac{1}{2h}\partial$  converge to the operators  $*, d$  and  $d^*$ , respectively. The algebra  $A$  intuitively becomes the algebra of differential forms with the usual wedge product and the discrepancy from Leibniz seen in finite differences disappears.

## 4 Computation of infinitesimal cumulants for $\delta$

In this section we apply the general construction of §2 to the cochain algebra  $(A, \wedge)$  and the square-zero degree 1 map  $\delta' \equiv \frac{1}{2h}\delta$ , explicitly calculating the infinitesimal cumulants  $\delta_k$ . We work in the algebra extending the scalars by the formal power series in the symbol  $h$ . At a given level  $i$  where the scale is  $2^{-i}$ , the operators written as quotients of operators on  $A$  by  $2h$  or  $h$ , as the case may be, mean actual numerical division by the values of  $h$  or  $2h$  at that scale, *e.g.* dividing by  $2h$  means dividing by  $2^{-i}$ .

**Theorem 4.1.** *The  $k$ -bracket (that is,  $(k-1)^{\text{th}}$  infinitesimal cumulant)  $\delta_k : S^k A \rightarrow A$  of  $(A, \wedge, \frac{1}{2h}\delta)$  can be decomposed in components  $\delta_k = \sum_{u=1}^n \delta_{k,u}$  where*

$$\delta_{k,u} = h^{k-1} m_k \circ \sum_{i=1}^k (-\Delta'_u \circ \bar{\text{id}})^{\otimes(i-1)} \otimes \delta_{1,u} \otimes \Delta_u^{\otimes(k-i)}$$

Here  $\Delta_u \equiv \frac{1}{h}(T'_u - \text{id})$  and  $\Delta'_u \equiv \frac{1}{h}(\text{id} - T_u)$  are the forward and backward divided half-differences, respectively, that is,  $(\Delta_u f)_I(p) = \frac{1}{h}(f_I(p + h\mathbf{e}_u) - f_I(p))$  and  $(\Delta'_u f)_I(p) = \frac{1}{h}(f_I(p) - f_I(p - h\mathbf{e}_u))$ . Meanwhile  $\delta_{1,u} = \frac{1}{2h}\delta_u$  is based on the symmetric divided difference,  $\delta_{1,u}(f dI) \equiv \frac{1}{2}(\Delta_u + \Delta'_u)f \cdot du \wedge dI$ . Finally,  $m_k$  denotes the iterated product  $A^{\otimes k} \rightarrow A$ .

*Proof.* The proof is by induction. For  $k=1$ , (8) gives  $\delta_{1,u} \equiv \frac{1}{2h}\delta_u$ . For the inductive step, assume the statement of the lemma for  $k-1$ . By Lemma 2.7,

$$\begin{aligned} \delta_{k,u}(v_1 \wedge v_2 \wedge v_3 \wedge \dots \wedge v_k) &= \delta_{k-1,u}(v_1 v_2 \wedge v_3 \wedge \dots \wedge v_k) \\ &\quad - (-1)^{|v_1|} v_1 \cdot \delta_{k-1,u}(v_2 \wedge v_3 \wedge \dots \wedge v_k) - (-1)^{|v_2|(1+|v_1|)} v_2 \cdot \delta_{k-1,u}(v_1 \wedge v_3 \wedge \dots \wedge v_k) \end{aligned}$$

Each term expands to a sum of  $k-1$  terms indexed by  $i=1, \dots, k-1$ . The contribution from  $i=1$  is

$$\begin{aligned} &\delta_{1,u}(v_1 v_2) - (-1)^{|v_1|} v_1 (\delta_{1,u} v_2) - (-1)^{|v_2|(1+|v_1|)} v_2 (\delta_{1,u} v_1) \\ &= (\delta_{1,u} v_1)(T'_u v_2) + (-1)^{|v_1|} (T_u v_1)(\delta_{1,u} v_2) - (-1)^{|v_1|} v_1 (\delta_{1,u} v_2) - (\delta_{1,u} v_1) v_2 \\ &= h(\delta_{1,u} v_1)(\Delta_u v_2) - h(-1)^{|v_1|} (\Delta'_u v_1)(\delta_{1,u} v_2) \end{aligned}$$

times  $h^{k-2}(m_{k-2} \circ \Delta_u^{\otimes(k-2)})(v_3 \otimes \dots \otimes v_k)$ , where in the first step Lemma 3.1(iii) was applied. This yields the  $i=1$  and  $i=2$  terms of the expression to be proved. The contribution from the three  $i$ -th terms (for  $i > 1$ ) is

$$\begin{aligned} &(-\Delta'_u \circ \bar{\text{id}})(v_1 v_2) - (-1)^{|v_1|} v_1 (-\Delta'_u \circ \bar{\text{id}})(v_2) - (-1)^{|v_2|(1+|v_1|)} v_2 (-\Delta'_u \circ \bar{\text{id}})(v_1) \\ &= (-1)^{|v_1|+|v_2|+1} (\Delta'_u(v_1 v_2) - v_1 \Delta'_u(v_2) - \Delta'_u(v_1) v_2) \\ &= (-1)^{|v_1|+|v_2|+1} \frac{1}{h} (v_1 v_2 - (T_u v_1)(T_u v_2) - v_1(v_2 - T_u v_2) - (v_1 - T_u v_1)v_2) \\ &= (-1)^{|v_1|+|v_2|} \frac{1}{h} (v_1 - T_u v_1)(v_2 - T_u v_2) \\ &= h(-1)^{|v_1|+|v_2|} (\Delta'_u v_1)(\Delta'_u v_2) = h(m \circ (-\Delta'_u \circ \bar{\text{id}})^{\otimes 2})(v_1 \otimes v_2) \end{aligned}$$



times  $h^{k-2}m_{k-2}((-\Delta'_u \circ \bar{\text{id}})^{\otimes(i-2)}(v_3 \otimes \cdots \otimes v_i) \otimes q_{1,u}v_{i+1} \otimes \Delta_u^{\otimes(k-1-i)}(v_{i+2} \otimes \cdots \otimes v_k))$ , which is the  $(i+1)$ -th term in the expression to be proved.  $\square$

*Remark 4.2.* In the limit  $h \rightarrow 0$ , both differences  $\Delta_u$  and  $\Delta'_u$  approach the partial derivative. The factor  $h^{k-1}$  in  $\delta_{k,u}$  ensures that  $\delta_{2,u} \rightarrow 0$ . Vanishing of the first infinitesimal cumulant is precisely the statement that Leibniz holds, in other words, the sequence of higher bracket structures,  $(A, \wedge, \frac{1}{2h}\delta)$  approximates a differential graded algebra, namely the differential graded algebra of differential forms  $(\Omega^*, \wedge, d)$  on the torus  $\mathbb{T}^n$ .

## 5 Computation of infinitesimal cumulants for $\partial$

In this section we apply the general construction of §2 to the chain algebra  $(C, \wedge)$  and the square-zero degree  $-1$  boundary map  $\frac{1}{2h}\partial$ , explicitly calculating the infinitesimal cumulants  $\partial_k$ .

Define the interior product  $i_u : C \rightarrow C$  by  $i_u(x) = *(*^{-1}x \wedge u)$ . Equivalently,  $(i_u(x))_I = x_{uI}$  while on monomials

$$i_u(f \cdot I) = \begin{cases} 0 & \text{if } u \notin I \\ \pm f \cdot (I \setminus \{u\}) & \text{if } u \in I \end{cases}$$

where the sign is defined so that for  $u \in I$ ,  $u \wedge i_u(I) = I$ .

**Lemma 5.1.** *The shifts and interior product satisfy*

- (i)  $(T_u i_u) \circ m = m \circ (T_u i_u \otimes T_u + T_u \bar{\text{id}} \otimes T_u i_u)$
- (ii)  $(T'_u i_u) \circ m = m \circ (T'_u i_u \otimes T'_u + T'_u \bar{\text{id}} \otimes T'_u i_u)$

*Proof.* The proofs of the two parts are identical. For (i), we compute (for  $u \notin I$ , otherwise both sides vanish),

$$\begin{aligned} ((T_u i_u)(f \cdot g))_I &= (T_u(f \cdot g))_{uI} \\ &= \sum_{J \cup K = uI} \pm_{JK}^u T_u(f_J g_K) \\ &= \sum_{J \cup K = I} (\pm_{uJK}^u T_u(f_u J g_K) + \pm_{JuK}^u T_u(f_J g_u K)) \\ &= \sum_{J \cup K = I} \pm_{JK} ((T_u f_u J)(T_u g_K) + (-1)^{|J|} (T_u f_J)(T_u g_u K)) \\ &= \sum_{J \cup K = I} \pm_{J,K} ((T_u i_u) f)_J (T_u g)_K + (-1)^{|J|} (T_u f)_J ((T_u i_u) g)_K \end{aligned}$$

as required. In the third line, the sum has been split according as  $u \in J$  or  $u \in K$  and the sets have been relabelled.  $\square$

**Theorem 5.2.** *The  $k$ -bracket (that is,  $(k-1)^{th}$  order infinitesimal cumulant) of  $(C, \wedge, \frac{1}{2\hbar}\partial)$  can be decomposed in components  $\partial_k = \sum_{u=1}^n \partial_{k,u}$  where*

$$\begin{aligned} \partial_{k,u} = & -\frac{1}{2}h^{k-2} \sum_{i=1}^k m_k \circ \left[ (\Delta_u \bar{\text{id}})^{\otimes(i-1)} \otimes (T'_u i_u) \otimes (\Delta_u)^{\otimes(k-i)} \right. \\ & \left. + (-1)^k (\Delta'_u \bar{\text{id}})^{\otimes(i-1)} \otimes (T_u i_u) \otimes (\Delta'_u)^{\otimes(k-i)} \right] \end{aligned}$$

where  $\Delta_u$  and  $\Delta'_u$  are the forward and backward divided half-differences, respectively, as in Theorem 4.1.

*Proof.* The proof proceeds by induction on  $k$ . For  $k=1$ , observe that by (7),  $\partial_u(f \cdot I) = (T_u f - T'_u f) \cdot i_u(I)$  and so  $\partial_{1,u} = \frac{1}{2\hbar}\partial_u = \frac{1}{2\hbar}(T_u - T'_u) \circ i_u$ . For the inductive step, assume the statement for  $k-1$ . By Lemma 2.7,

$$\begin{aligned} \partial_{k,u}(v_1 \wedge v_2 \wedge v_3 \wedge \dots \wedge v_k) &= \partial_{k-1,u}(v_1 v_2 \wedge v_3 \wedge \dots \wedge v_k) \\ &- (-1)^{|v_1|} v_1 \cdot \partial_{k-1,u}(v_2 \wedge v_3 \wedge \dots \wedge v_k) - (-1)^{|v_2|(1+|v_1|)} v_2 \cdot \partial_{k-1,u}(v_1 \wedge v_3 \wedge \dots \wedge v_k) \end{aligned}$$

Each term expands to a sum of  $2(k-1)$  terms, two from each  $i=1, \dots, k-1$ . The first terms for  $i=1$  contribute

$$\begin{aligned} & (T'_u i_u)(v_1 v_2) - (-1)^{|v_1|} v_1 (T'_u i_u v_2) - (-1)^{|v_2|(1+|v_1|)} v_2 (T'_u i_u v_1) \\ &= (T'_u i_u v_1)(T'_u v_2) + (T'_u \bar{\text{id}} v_1)(T'_u i_u v_2) - (-1)^{|v_1|} v_1 (T'_u i_u v_2) - (T'_u i_u v_1) v_2 \\ &= h(T'_u i_u v_1)(\Delta_u v_2) + h(-1)^{|v_1|} (\Delta_u \bar{\text{id}} v_1)(T'_u i_u v_2) \end{aligned}$$

times  $-\frac{1}{2}h^{k-3}(m_{k-2} \circ \Delta_u^{\otimes(k-2)})(v_3 \otimes \dots \otimes v_k)$ . Here we used Lemma 5.1(ii) in the first line. These are the evaluations of the first term in the sum for  $\partial_{k,u}$  to be proved, when  $i=2, i=1$  respectively. The second terms for  $i=1$  contribute similarly, only there is an extra sign,  $T_u$  replaces  $T'_u$  and  $\Delta'_u$  replaces  $\Delta_u$ .

The contribution to  $\partial_{k,u}(v_1 \wedge \dots \wedge v_k)$  from the  $i$ -th terms of the second type is

$$\begin{aligned} & (\Delta'_u \bar{\text{id}})(v_1 v_2) - (-1)^{|v_1|} v_1 (\Delta'_u \bar{\text{id}})(v_2) - (-1)^{|v_2|(|v_1|+1)} v_2 (\Delta'_u \bar{\text{id}})(v_1) \\ &= -h(m \circ (\Delta'_u \bar{\text{id}})^{\otimes 2})(v_1 \otimes v_2) \end{aligned}$$

times  $-\frac{1}{2}(-h)^{k-3} m((\Delta_u \bar{\text{id}})^{\otimes(i-2)}(v_3 \otimes \dots \otimes v_i) \otimes (T'_u i_u v_{i+1}) \otimes (\Delta_u)^{\otimes(k-i-1)}(v_{i+2} \otimes \dots \otimes v_k))$ . The equality follows by the argument in the last block of calculation in the proof of Theorem 4.1. The resulting expression is the contribution of the second type of term for  $i+1$  in the expression to be proved. A similar argument goes for the first type of term, only with a different sign and  $\Delta_u$  in place of  $\Delta'_u$ .  $\square$

*Example 5.3.* Here are the formulae for the first few brackets for  $n=3$ . Write the  $(k-1)^{th}$  infinitesimal cumulant  $\partial_k(v_1 \wedge \dots \wedge v_k)$  instead, using bracket

notation, as  $[v_1, \dots, v_k]$ . Then the 2-bracket on chains splits into pieces  $[\cdot, \cdot] = [\cdot, \cdot]_x + [\cdot, \cdot]_y + [\cdot, \cdot]_z$  (this is the splitting of  $\partial_k$  into pieces  $\partial_{k,u}$ ) where

$$[f \cdot I, g \cdot J]_u = \begin{cases} \frac{1}{2}((\Delta_u f + \bar{\Delta}_u f)g - f(\Delta_u g + \bar{\Delta}_u g))i_u(I) \cdot J & \text{if } u \in I, J \\ -\frac{1}{2}((T_u f)(\bar{\Delta}_u g) + (T'_u f)(\Delta_u g)) \cdot i_u(I \cdot J) & \text{if } u \in I, u \notin J \\ -\frac{1}{2}((\bar{\Delta}_u f)(T_u g) + (\Delta_u f)(T'_u g)) \cdot i_u(I \cdot J) & \text{if } u \notin I, u \in J \\ 0 & \text{if } u \notin I, J \end{cases}$$

The 3-bracket  $[\cdot, \cdot, \cdot]$  on chains similarly splits into three parts as  $[\cdot, \cdot, \cdot]_x + [\cdot, \cdot, \cdot]_y + [\cdot, \cdot, \cdot]_z$  where  $[f \cdot I, g \cdot J, k \cdot K]_u$  is  $\frac{h}{2}$  times:

$$\begin{array}{ll} 0 & \text{if } u \in I, J, K \\ (-1)^{|I|}[(f\Delta_u g - g\Delta_u f)\Delta_u k + (g\bar{\Delta}_u f - f\bar{\Delta}_u g)\bar{\Delta}_u k] \cdot Ii_u(J)K & \text{if } u \in I, J, K^c \\ (-1)^{|I|+|J|}[(f\Delta_u k - k\Delta_u f)\Delta_u g + (k\bar{\Delta}_u f - f\bar{\Delta}_u k)\bar{\Delta}_u g] \cdot IJi_u(K) & \text{if } u \in I, J^c, K \\ (-1)^{|J|}[(k\Delta_u g - g\Delta_u k)\Delta_u f + (g\bar{\Delta}_u k - k\bar{\Delta}_u g)\bar{\Delta}_u f] \cdot Ii_u(J)K & \text{if } u \in I^c, J, K \\ (T_u f \cdot \bar{\Delta}_u g \cdot \bar{\Delta}_u k - T'_u f \cdot \Delta_u g \cdot \Delta_u k) \cdot i_u(IJK) & \text{if } u \in I, J^c, K^c \\ (\bar{\Delta}_u f \cdot T_u g \cdot \bar{\Delta}_u k - \Delta_u f \cdot T'_u g \cdot \Delta_u k) \cdot i_u(IJK) & \text{if } u \in I^c, J, K^c \\ (\bar{\Delta}_u f \cdot \bar{\Delta}_u g \cdot T_u k - \Delta_u f \cdot \Delta_u g \cdot T'_u k) \cdot i_u(IJK) & \text{if } u \in I^c, J^c, K \\ 0 & \text{if } u \notin I, J, K \end{array}$$

*Remark 5.4.* In the limit  $h \rightarrow 0$ , the differences  $\Delta_u$  and  $\Delta'_u$  approach the partial derivative. The factor  $h^{k-2}$  in  $\partial_{k,u}$  ensures that  $\partial_{3,u} \rightarrow 0$  although  $\partial_{2,u}$  does not approach zero. Vanishing of the 3-bracket (second infinitesimal cumulant) in the continuum limit is the second order derivation property, along with a non-trivial structure  $\partial_2 = [\partial, \wedge]$ , namely the Schouten-Nijenhuis bracket on multivector fields, the natural extension of the Lie bracket on ordinary vector fields.

## 6 Binary QFT Algebras

We use an adaptation to our circumstances of the definition of binary QFT algebras from [2] Definition 2.3.

**Definition 6.1.** *If a (graded) commutative associative algebra over  $\mathbb{Q}[[h]]$  has a differential, this determines a structure consisting of all the brackets, Definition 2.4. (Known to be expressible in terms of commutators of the differential with the iterated products.) If these all vanish this is called a classical algebra. If for all  $k \in \mathbb{N}$ , the  $k^{\text{th}}$  higher infinitesimal cumulant (that is, the  $(k+1)$ -bracket) is divisible by  $h^k$ , this is called a binary QFT algebra.*

**Definition 6.2.** *The sequence of  $k$ -brackets on  $V$  associated to the structure of a binary QFT algebra on a graded linear space  $V$  over  $\mathbb{Q}[[h]]$  determines a coderivation  $D$  of  $SV$ , of square zero. A coalgebra mapping  $I : SV \rightarrow SW$ , preserving the monomial filtration which intertwines the respective coderivations*

$D$  on  $SV$  and  $D$  on  $SW$ , is a (nonlinear) morphism of binary QFT structures on  $V$  and  $W$  if its  $k^{\text{th}}$  Taylor coefficient  $S^k V \rightarrow W$  is divisible by  $h^{k-1}$ .

**Discussion of  $\partial$  and the chain algebra.** Take the lattice chain algebra  $C$  of §3 and extend the scalars to  $\mathbb{Q}[[h]]$ . Furthermore we add to these the difference quotient operators,  $\Delta_u$  (see the statement of Theorem 4.1 for their definition), putting in the relation

$$T'_u c - c = h\Delta_u c$$

for all chains  $c$ , to make the difference between a chain and its translate by  $h$  in any lattice direction formally divisible by  $h$ . The resulting chain algebra we will denote by  $C(h)$ .

On this algebra there are unary operations which reflect the geometry of the chain algebra:  $T_u$  (translation through  $h$  in the  $u^{\text{th}}$  lattice direction, see §3),  $\bar{\text{id}}$  (introducing a sign  $(-1)^c$  for a  $c$ -chain; see §2.3),  $i_u$  (interior product of the chain with the  $u$ -direction; see the beginning of §5 for its definition) along with  $\Delta_u$ . All compositions of iterated multiplications  $m_k : S^{k+1}C(h) \rightarrow C(h)$  with the unary operators  $T_u$ ,  $\bar{\text{id}}$ ,  $i_u$ ,  $\Delta_u$ , and their linear combinations, are considered as allowed morphisms.

Recall from the proof of Theorem 4.1 that  $\Delta'_u = \Delta_u \circ T_u$  while from the proof of Theorem 5.2,  $\partial = \sum_u (T_u - T'_u) \circ i_u = -h \sum_u (\Delta_u + \Delta'_u) \circ i_u$ . This makes  $\partial$  divisible by  $h$  as an operator on  $C(h)$  and so we define  $\partial' \equiv \frac{1}{h}\partial$ . We can now state the formalisation of Theorem 1.2(A).

**Theorem 6.3.** *The triple  $(C(h), \wedge, \partial)$  defines a binary QFT algebra.*

*Proof.* By Theorem 5.2, the  $k^{\text{th}}$  infinitesimal cumulant (that is the  $(k+1)$ -bracket) of  $(C(h), \wedge, \partial')$  is divisible by  $h^{k-1}$  and so according to the above definition,  $(C(h), \wedge, h\partial')$  defines a binary QFT.  $\square$

**Discussion of  $\delta$  and the cochain algebra.** Now we take the lattice cochain algebra  $A$  of §3 and extend the scalars to  $\mathbb{Q}[[h]]$ , adjoin difference quotients and add the relation

$$T'_u a - a = h\Delta_u a$$

for all cochains  $a$ , to make the difference between a cochain and its translate by  $h$  in any lattice direction formally divisible by  $h$ . The resulting cochain algebra we will denote by  $A(h)$ .

On this algebra there are unary operations which reflect the geometry of the cochain algebra:  $T_u$ ,  $\bar{\text{id}}$  and wedge with  $du$ , along with  $\Delta_u$ . Furthermore, by (8) in §3,  $\delta(fdI) = \sum_u (T'_u f - T_u f) du \wedge dI = h \sum_u du \wedge (\Delta_u + \Delta'_u)(fdI)$  is divisible by  $h$  and so we can define the operator  $\delta' \equiv \frac{1}{2h}\delta$  on  $A(h)$ .

The formalisation of Theorem 1.3(A) is as follows.

**Theorem 6.4.** *The triple  $(A(h), \wedge, \delta')$  defines a binary QFT algebra.*

*Proof.* By Theorem 4.1, the  $k^{\text{th}}$  infinitesimal cumulant (that is the  $(k + 1)$ -bracket) of  $(A(h), \wedge, \delta')$  is divisible by  $h^k$  as required.  $\square$

*Remark 6.5.* While  $\delta, \partial$  are the familiar boundary and coboundary operators of algebraic topology on chains and cochains respectively, the operators  $\delta'$  and  $\partial'$  are the ones which will give first order discrete approximations to the continuum calculus.

## 7 Maps between scales

### 7.1 Coarse scale lattice inside finer lattice

Contained in the lattice  $L_h$  is the sublattice  $L_{2h} = (2h\mathbb{Z})^n / (4Nh\mathbb{Z})^n$ , a periodic cubic lattice with grid step  $2h$  and size  $2N$  in each direction. The vertices of  $L_{2h}$  are bicoloured similarly to  $L_h$  by the parity of  $\frac{1}{2h} \sum_u x_u$ , although it should be noted that the colourings of the two lattices are *not* compatible. Denote the corresponding constructions to those of §3 on the coarser lattice by a bar,  $\bar{I}_a$  being a cell of type  $I$  with edge lengths  $4h$  and centre  $a \in L_{2h}$ ,  $\bar{I}a \equiv IIa$  denoting a shift of  $a$  forward by  $2h$  in each of the directions in  $I$ , and so forth, culminating in the coarse chain space  $\bar{C}$  with its pointwise multiplication  $\wedge$ .

### 7.2 Chain and cochain maps

There is a natural map from a chain complex of a coarse lattice to that of a subdivision called *crumbling*. The dual map on cochains is *integration*, from cochains on the fine lattice to cochains on a coarser lattice. However there is a subtlety in our case, in that our chains are generated by **all** cubes of edge length  $2h$ , so that there are finer  $k$ -cubes which do not appear in subdivisions of coarser  $k$ -cubes.

To analyze this for  $n = 3$ , the chain complexes we are studying at each scale are naturally the direct sum of 8 copies of shift-isomorphic subcomplexes. Cutting the scale  $h$  in half also yields 8 direct summands of the finer chain complex. The finer 8 can be respectively shifted to become subdivisions of the coarser 8 summands and then the crumbling chain mappings (and their dual, the integration maps on cochains) are present. However, the shifts that effect this picture are not unique.

To give precise formulae, note that  $C = E^{\otimes n}$  where  $E$  is the chain complex of a one-dimensional lattice  $h\mathbb{Z}/4Nh\mathbb{Z}$ , whose chains are generated by points  $\emptyset_a$  and intervals of length  $2h$ , the one centred at  $a$  being denoted  $x_a$ . On  $E$ , the boundary acts  $\partial : E_1 \rightarrow E_0$ ,

$$\partial(x_a) = \emptyset_{a+h} - \emptyset_{a-h}$$

In this one-dimensional setting, the chain complex splits into a direct sum of two copies, those with even vertices and those with odd vertices. Choose the shifts

on the finer chain complex so as to be trivial on the even vertex subcomplex, and to be a shift to the right by  $h$  on the odd vertex subcomplex. The corresponding crumbling map  $\iota : \bar{E} \rightarrow E$  is given by

$$\iota(\bar{\theta}_a) = \begin{cases} \theta_a & \text{if } a \in 4h\mathbb{Z} \\ \theta_{a-h} & \text{if } a \in 2h + 4h\mathbb{Z} \end{cases} \quad \text{and } \iota(\bar{x}_a) = \begin{cases} x_a + x_{a-2h} & \text{if } a \in 4h\mathbb{Z} \\ x_{a-h} + x_{a+h} & \text{if } a \in 2h + 4h\mathbb{Z} \end{cases}$$

**Lemma 7.1.**  $\iota : \bar{E} \rightarrow E$  is a chain map for  $\partial$ , that is  $\iota \circ \bar{\partial} = \partial \circ \iota$ .

Tensoring  $n$  times,  $\iota$  induces a chain map  $\iota : (\bar{C}, \bar{\partial}) \rightarrow (C, \partial)$ .

Similarly for cochains, identify the space of cochains on the  $n$ -dimensional lattice as  $A = B^{\otimes n}$  where  $B$  is the cochain complex of the one-dimensional lattice, consisting of 0-cochains  $f(x)$  and 1-cochains  $f(x)dx$  (which takes value  $f(x)$  on the interval whose centre is  $x$ ), using the notation of §3. On  $B$ , the coboundary acts  $\delta : B_0 \rightarrow B_1$  with

$$\delta f = (f(x+h) - f(x-h)) dx$$

The integration map  $\bar{-} : B \rightarrow \bar{B}$  dual to the above crumbling map on chains is given on 0-cochains by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in 4h\mathbb{Z} \\ f(x-h) & \text{if } x \in 2h + 4h\mathbb{Z} \end{cases}$$

while on 1-cochains,  $f(x)dx \mapsto g(x)dx$  where

$$g(x) = \begin{cases} \frac{1}{2}(f(x) + f(x-2h)) & \text{if } x \in 4h\mathbb{Z} \\ \frac{1}{2}(f(x-h) + f(x+h)) & \text{if } x \in 2h + 4h\mathbb{Z} \end{cases}$$

Tensoring  $n$  times gives a cochain map  $\bar{-} : (A, h^{-1}\delta) \rightarrow (\bar{A}, (2h)^{-1}\bar{\delta})$ , that is  $\frac{1}{2}\bar{\delta} \circ \bar{-} = \bar{-} \circ \delta$ .

### 7.3 General theory of morphisms extending chain maps

Suppose now that  $(V, m)$  and  $(\bar{V}, \bar{m})$  are two commutative associative algebras, each endowed with a square-zero map of (the same) grading  $\pm 1$ ,  $\partial : V \rightarrow V$  and  $\bar{\partial} : \bar{V} \rightarrow \bar{V}$ . Suppose that  $\bar{-} : V \rightarrow \bar{V}$  is a chain map respecting the grading, that is  $\bar{\partial}\bar{x} = \overline{(\partial x)}$  for  $x \in V$ . Note that there is no assumption about the compatibility or otherwise of  $\bar{-}$  with the algebra structures on  $V$  and  $\bar{V}$ .

Following the construction of Theorem 2.10, we generate coderivations  $D$  and  $\bar{D}$  on  $S^*V$  and  $S^*\bar{V}$  respectively, each of square-zero, whose Taylor coefficients describe the infinitesimal cumulants of  $\partial$  and  $\bar{\partial}$ , with respect to  $m$  and  $\bar{m}$ , respectively. Let  $S(\bar{-}) : S^*V \rightarrow S^*\bar{V}$  be the image of  $\bar{-}$  under the functor  $S$  from chain complexes to differential coalgebras; that is

$$S(\bar{-}) : v_1 \wedge \dots \wedge v_k \mapsto \bar{v}_1 \wedge \dots \wedge \bar{v}_k$$

This is a coalgebra map and it follows from the fact that  $\bar{\cdot}$  is a chain map, that

$$S(-) \circ \partial^\wedge = \bar{\partial}^\wedge \circ S(-)$$

Using the cumulant bijections  $\tau : S^*V \rightarrow S^*V$  and  $\bar{\tau} : S^*\bar{V} \rightarrow S^*\bar{V}$  which are coalgebra maps, the composition

$$\sigma \equiv \bar{\tau}^{-1} \circ S(-) \circ \tau : S^*V \rightarrow S^*\bar{V}$$

is seen to be a coalgebra map of degree 0.

**Lemma 7.2.** *The map  $\sigma : S^*V \rightarrow S^*\bar{V}$  defines a coalgebra morphism with  $\sigma \circ D = \bar{D} \circ \sigma$ .*

The proof is the commuting cube diagram below.

$$\begin{array}{ccccc}
S^*V & \xrightarrow{S(-)} & S^*\bar{V} & & \\
\uparrow \tau & \searrow \partial^\wedge & \uparrow \bar{\partial}^\wedge & & \\
& S^*V & \xrightarrow{S(-)} & S^*\bar{V} & \\
& \uparrow \tau & \uparrow \bar{\tau} & & \\
S^*V & \xrightarrow{\sigma} & S^*\bar{V} & & \\
\downarrow D & & \downarrow \bar{D} & & \\
& S^*V & \xrightarrow{\sigma} & S^*\bar{V} & \\
& \uparrow \tau & \uparrow \bar{\tau} & & \\
& S^*V & \xrightarrow{\sigma} & S^*\bar{V} & 
\end{array}$$

As with any coalgebra map,  $\sigma$  is determined by its Taylor coefficients, which we denote  $\sigma_k : S^kV \rightarrow \bar{V}$ . We will now give an explicit formula for the action of  $\sigma_k$  along with a recursion relation, which will use in the proof of estimates in §7.4. These will be analogues of Lemma 2.6 and Lemma 2.7.

**Lemma 7.3.**

$$\sigma_k(v_1 \wedge \cdots \wedge v_k) = \sum_{r=1}^k (-1)^{r-1} (r-1)! \sum_{I_1 \cup \cdots \cup I_r = [k]} \pm \bar{\tau}_1(\overline{\tau_1(v_{I_1})} \wedge \cdots \wedge \overline{\tau_1(v_{I_r})})$$

where the inner summation is over all (unordered) partitions of  $[k]$  into  $r$  subsets, the sign is that determined by the change of order of symbols  $v_1 \wedge \cdots \wedge v_k = \pm v_{I_1} \wedge \cdots \wedge v_{I_r}$ .

*Proof.* By definition,  $\sigma = \bar{\tau}^{-1} \circ S(-) \circ \tau$ . Its  $k^{\text{th}}$  Taylor coefficient is  $\sigma_k = (p_1 \circ \bar{\tau}^{-1} \circ S(-) \circ \tau)|_{S^kV}$  which we compute from the first part of Lemma 2.3,

$$\sigma_k(v_1 \wedge \cdots \wedge v_k) = (p_1 \circ \bar{\tau}^{-1} \circ S(-)) \sum_{r=1}^k \sum_{I_1 \cup \cdots \cup I_r = [k]} \pm \tau_1(v_{I_1}) \wedge \cdots \wedge \tau_1(v_{I_r})$$

where the sum is over all (unordered) partitions of  $[k]$  into  $r$  non-empty subsets, and the sign is as in the statement of the lemma. Recalling the definition of  $S(-)$  and applying the equality  $p_1 \circ \bar{\tau}^{-1} = d_r \cdot \bar{\tau}_1$  on  $S^rV$  (Lemma 2.3), the statement of the lemma follows.  $\square$

*Remark 7.4.* The last lemma identifies  $\sigma_k$  with the  $k^{\text{th}}$  (commutative) cumulant of  $\bar{\cdot}$  with respect to the multiplications  $m$  and  $\bar{m}$ ; see [4]. The first few are

$$\begin{aligned}\sigma_1(u) &= \bar{u} \\ \sigma_2(u \wedge v) &= \bar{u}\bar{v} - \bar{u} \cdot \bar{v} \\ \sigma_3(u \wedge v \wedge w) &= \bar{u}\bar{v}\bar{w} - \bar{u}\bar{v} \cdot \bar{w} - \bar{u} \cdot \bar{v}\bar{w} - (-1)^{|u| \cdot |v|} \bar{v} \cdot \bar{u}\bar{w} + 2\bar{u} \cdot \bar{v} \cdot \bar{w}\end{aligned}$$

**Lemma 7.5.** For  $u, v \in V$  and  $w \in S^k V$ ,

$$\sigma_{k+2}(u \wedge v \wedge w) = \sigma_{k+1}(uv \wedge w) - \sum_{J \subseteq [k]} \pm \sigma_{|J|+1}(u \wedge w_J) \sigma_{|J^c|+1}(v \wedge w_{J^c})$$

where the sum is over all (not necessarily proper) subsets  $J$  of  $[k]$  and the sign is so that  $w_J \wedge v \wedge w_{J^c} = \pm v \wedge w$ .

*Proof.* By Lemma 7.3, the left hand side can be written as a sum of terms indexed by partitions of  $\{u, v\} \cup [k]$ .

If  $u$  and  $v$  are in the same set in the partition then the corresponding term in  $\sigma_{k+2}(u \wedge v \wedge w)$  will also appear in  $\sigma_{k+1}(uv \wedge w)$  and with the same coefficient.

On the other hand, if  $u$  and  $v$  are in different sets in the partition  $I_1 \cup \dots \cup I_r$ , say  $u \in I_1$  and  $v \in I_2$ , then the corresponding term in the LHS of the equality in the lemma may appear several times in the sum on the RHS. In particular, it will appear in the term on the RHS labelled by the set  $J \subseteq [k]$  if  $\{u\} \cup J$  can be written as a union of some collection of the sets  $I_j$  (while  $\{v\} \cup J^c$  is the union of the complementary collection), say  $\cup_{j \in K} I_j$  for some  $K \subseteq [r]$  where  $1 \in K$  and  $2 \notin K$ . The coefficient of the term is

$$- \sum_{K \subseteq [r], 1 \in K, 2 \notin K} d_{|K|} d_{r-|K|} = (-1)^{r-1} \sum_{m=1}^{r-1} \binom{r-2}{m-1} (m-1)! (r-m-1)! = d_r$$

which matches the coefficient of the same term in  $\sigma_{k+2}(u \wedge v \wedge w)$   $\square$

**Lemma 7.6.** Suppose that  $V, W, \bar{V}$  and  $\bar{W}$  are (graded) commutative associative algebras with given grading preserving maps  $\bar{\cdot} : V \rightarrow \bar{V}$  and  $\bar{\cdot} : W \rightarrow \bar{W}$ . Then their tensor product  $\bar{\cdot} : V \otimes W \rightarrow \bar{V} \otimes \bar{W}$  has  $k^{\text{th}}$  commutative cumulant  $S^k(V \otimes W) \rightarrow \bar{V} \otimes \bar{W}$  given by

$$\begin{aligned}(v_1 \otimes w_1) \wedge \dots \wedge (v_k \otimes w_k) \\ \mapsto \sum_{r=1}^k \sum_{I_1 \cup \dots \cup I_r = [k]} \pm \sigma_{|I_1|}^V(v_{I_1}) \dots \sigma_{|I_r|}^V(v_{I_r}) \otimes \sigma_r^W(\tau_1(w_{I_1}) \wedge \dots \wedge \tau_1(w_{I_r}))\end{aligned}$$

in terms of the cumulants  $\sigma_k^V : S^k V \rightarrow \bar{V}$  and  $\sigma_k^W : S^k W \rightarrow \bar{W}$ . Here the inner sum is over all unordered partitions of  $[k]$  into  $r$  non-empty sets and the sign is that induced by the change of order

$$v_1 \wedge w_1 \wedge \dots \wedge v_k \wedge w_k = \pm v_{I_1} \wedge \dots \wedge v_{I_r} \wedge w_{I_1} \wedge \dots \wedge w_{I_r}$$



*Proof.* Unordered partitions  $\pi$  of  $[k]$  are in bijection with equivalence relations  $\sim$  on the set  $[k]$ . There is a partial ordering on equivalence relations by which  $\sim'$  is said to be *stronger* than  $\sim$  if whenever  $a \sim' b$  then also  $a \sim b$ . The corresponding partial ordering on (unordered) partitions will be written  $\pi' \geq \pi$  and we say that  $\pi'$  is a *finer* partition than  $\pi$  (equivalently,  $\pi$  is said to be *coarser* than  $\pi'$ , written  $\pi \leq \pi'$ ).

For a partition  $\pi$  of  $[k]$  represented by  $I_1 \cup \dots \cup I_r = [k]$ , denote by  $|\pi|$ , the number of equivalence classes (that is,  $r$ ). Denote by  $v_\pi \in V$ , the element  $\bar{\tau}_1(\tau_1(v_{I_1}) \wedge \dots \wedge \tau_1(v_{I_r}))$ . In this notation, Lemma 7.3 can be written,

$$\sigma_k(v_1 \wedge \dots \wedge v_k) = \sum_{\pi} d_{|\pi|} v_\pi$$

Using Lemma 7.3 to expand out the right hand side of the expression to be proved, we obtain

$$\sum_{\pi} \sum_{\pi' \geq \pi} \pm \left( \prod_{j=1}^{|\pi|} d_{\alpha_j} \right) v_{\pi'} \otimes \sum_{\pi'' \leq \pi} d_{|\pi''|} w_{\pi''} \quad (*)$$

where the sums are over all partitions  $\pi$  of  $[k]$ , finer partitions  $\pi'$  and coarser partitions  $\pi''$  and  $\alpha_j$  ( $1 \leq j \leq |\pi|$ ) is the number of sets into which  $\pi'$  splits the  $j^{\text{th}}$  set of  $\pi$ . The coefficient of a term  $v_{\pi'} \otimes w_{\pi''}$  for partitions  $\pi'$  and  $\pi''$  of  $[k]$  with  $\pi' \geq \pi''$  is

$$\pm d_{|\pi''|} \sum_{\pi: \pi'' \leq \pi \leq \pi'} \prod_{j=1}^{|\pi|} d_{\alpha_j}$$

where the sign is given by the change of order of  $v$ 's and  $w$ 's (which is independent of  $\pi$ ). If  $\beta_i$  ( $1 \leq i \leq |\pi''|$ ) denotes the number of sets into which  $\pi'$  splits the  $i^{\text{th}}$  set of  $\pi''$  then  $\pi$  is determined by a choice of partition of  $[\beta_i]$  for each  $i$ . The above sum then splits as a product,

$$\pm d_{|\pi''|} \prod_{i=1}^{|\pi''|} \left( \sum_{J_1 \cup \dots \cup J_s = [\beta_i]} \prod_{l=1}^s d_{|J_l|} \right)$$

By the last statement in the proof of Lemma 2.3, the inner sum vanishes unless  $\beta_i = 1$  and then it is 1. The whole expression therefore vanishes unless  $\beta_i = 1$  for all  $i$ , that is,  $\pi' = \pi''$  and then is  $\pm d_{|\pi''|}$ . The expression (\*) therefore reduces to

$$\sum_{\pi''} \pm d_{|\pi''|} v_{\pi''} \otimes w_{\pi''}$$

which is, by Lemma 7.3, the  $k^{\text{th}}$  cumulant of the tensor product, as required.  $\square$

## 7.4 Application to refinement of cubical lattices

The construction of §7.3 now induces from the cochain map  $\bar{\tau}$  of §7.2, a corresponding extended morphism from fine to coarse, which intertwines the coderivations  $D$  and  $\bar{D}$  on the symmetric cofree coalgebras of cochains. That is, they

‘intertwine’ the collection of infinitesimal cumulants from the finer and coarser lattices, although the exact relations between individual coarse and fine infinitesimal cumulants are not so simple, see Remark 7.9 below.

**Lemma 7.7.** *For  $k > 1$ , the cumulant  $\sigma_k : S^k B \rightarrow \bar{B}$  of the map  $\bar{\phantom{x}}$  defined in §7.2 on the one-dimensional lattice, is divisible by  $h^{k-1}$ , that is, the components of the cochain  $\sigma_k(v_1 \wedge \cdots \wedge v_k) \in \bar{B}$  (with respect to the natural basis) can be written as sums of terms, each of which contains a product of at least  $k-1$  factors of the form of a difference in values of components of the cochains  $v_1, \dots, v_k$  at lattice points  $2h$  apart.*

*Proof.* By Lemma 7.3,

$$\sigma_k(v_1 \wedge \cdots \wedge v_k) = \sum_{r=1}^k \frac{(-1)^{r-1}}{r} \sum_{I_1 \cup \cdots \cup I_r = [k]} \pm \bar{\tau}_1 \left( \overline{\tau_1(v_{I_1})} \wedge \cdots \wedge \overline{\tau_1(v_{I_r})} \right)$$

where the sum is over ordered partitions  $I_1 \cup \cdots \cup I_r$  of  $[k]$  into  $r$  non-empty subsets. Now  $\bar{B}$  has a basis consisting of 0-cochains localized at a single point,  $\bar{\chi}_a$ , and 1-cochains localized on a single interval (centre  $a$ ) of length  $4h$ ,  $\bar{\chi}_a dx$ , where  $a$  is a multiple of  $2h$ . Since the formulae for  $\bar{\phantom{x}}$  are invariant under shifts by  $4h$ , it suffices to compute the coefficients of  $\bar{\chi}_0$ ,  $\bar{\chi}_{2h}$ ,  $\bar{\chi}_0 dx$  and  $\bar{\chi}_{2h} dx$ . For example, to compute the component of  $\bar{\chi}_0$  in  $\sigma_k(v_1 \wedge \cdots \wedge v_k)$ , note that

$$[\bar{\tau}_1(w_1 \wedge \cdots \wedge w_r)]_{\bar{\chi}_0} = \prod_{j=1}^r [w_j]_{\bar{\chi}_0}, \quad [\bar{v}]_{\bar{\chi}_0} = [v]_{\chi_0}, \quad [\tau_1(v_1 \wedge \cdots \wedge v_s)]_{\chi_0} = \prod_{j=1}^s [v_j]_{\chi_0}$$

from which we obtain the following expression for  $[\sigma_k(v_1 \wedge \cdots \wedge v_k)]_{\bar{\chi}_0}$ ,

$$\sum_{r=1}^k \frac{(-1)^{r-1}}{r} \sum_{I_1 \cup \cdots \cup I_r = [k]} \prod_{j=1}^r \left( \prod_{i \in I_j} [v_i]_{\chi_0} \right) = \left( \prod_{i=1}^k [v_i]_{\chi_0} \right) \cdot \sum_{r=1}^k \frac{(-1)^{r-1}}{r} N_{k,r} = 0 \quad (10)$$

the latter equality following by a proof similar to that of (3) in §2.3. Similarly  $[\sigma_k(v_1 \wedge \cdots \wedge v_k)]_{\bar{\chi}_{2h}}$  vanishes for  $k > 1$ . To complete the proof of the lemma, we need to check the divisibility by  $h^{k-1}$  of components of  $\bar{\chi}_0 dx$  and  $\bar{\chi}_{2h} dx$  in  $\sigma_k(v_1 \wedge \cdots \wedge v_k)$ . Note that

$$\begin{aligned} [\bar{\tau}_1(w_1 \wedge \cdots \wedge w_r)]_{\bar{\chi}_0 dx} &= \sum_{p=1}^r \left( [w_p]_{\bar{\chi}_0 dx} \prod_{\substack{j=1 \\ j \neq p}}^r [w_j]_{\bar{\chi}_0} \right) \\ [\bar{v}]_{\bar{\chi}_0} &= [v]_{\chi_0}, \quad [\bar{v}]_{\bar{\chi}_0 dx} = \frac{1}{2} [v]_{\chi_0 dx} + \frac{1}{2} [v]_{\chi_{-2h} dx} \\ [\tau_1(v_1 \wedge \cdots \wedge v_s)]_{\chi_0} &= \prod_{j=1}^s [v_j]_{\chi_0}, \quad [\tau_1(v_1 \wedge \cdots \wedge v_r)]_{\chi_a dx} = \sum_{p=1}^r \left( [v_p]_{\chi_a dx} \prod_{\substack{j=1 \\ j \neq p}}^r [v_j]_{\chi_a} \right) \end{aligned}$$

from which we obtain the following expression for  $[\sigma_k(v_1 \wedge \cdots \wedge v_k)]_{\bar{\chi}_0 dx}$ ,

$$\sum_{r=1}^k \frac{(-1)^{r-1}}{r} \sum_{I_1 \cup \cdots \cup I_r = [k]} \sum_{p=1}^r \frac{1}{2} \prod_{\substack{j=1 \\ j \neq p}}^r \left( \prod_{i \in I_j} [v_i]_{\chi_0} \right) \cdot \left( \sum_{q \in I_p} ([v_q]_{\chi_0} dx \prod_{i \in I_p \setminus \{q\}} [v_i]_{\chi_0} + [v_q]_{\chi_{-2h} dx} \prod_{i \in I_p \setminus \{q\}} [v_i]_{\chi_{-2h}}) \right) \quad (11)$$

This is totally symmetric under permutation of the  $v_i$ 's and depends linearly on the coefficients of the localized 1-cochains  $\chi_0 dx$  and  $\chi_{-2h} dx$  in  $v_1, \dots, v_k$ . In particular, the coefficient of  $[v_k]_{\chi_0 dx}$  in (11) vanishes, reducing to a sum of a form similar to (10). Meanwhile, the coefficient of  $[v_k]_{\chi_{-2h} dx}$  in (11) is

$$\sum_{r=1}^k \frac{(-1)^{r-1}}{2r} \sum_{p=1}^r \sum_{\substack{I_1 \cup \cdots \cup I_r = [k] \\ k \in I_p}} \prod_{i \in I_p \setminus k} [v_i]_{\chi_{-2h}} \cdot \prod_{i \in \cup_{j \neq p} I_j} [v_i]_{\chi_0}$$

Noting that the internal summand depends on the partition  $\{I_j\}$  only via  $I_p$ , we can rewrite it as  $\sum_{r=1}^k \frac{1}{2} (-1)^{r-1} \sum_{k \in I \subseteq [k]} N_{k-|I|, r-1} \prod_{i \in I \setminus k} [v_i]_{\chi_{-2h}} \cdot \prod_{i \in I^c} [v_i]_{\chi_0}$ , which can be identified as  $\frac{1}{2} \left( \frac{\partial}{\partial [v_k]_{\chi_{-2h}}} \right)$  of the expression

$$\sum_{r=1}^k (-1)^{r-1} \sum_{I \subseteq [k]} N_{k-|I|, r-1} \prod_{i \in I} [v_i]_{\chi_{-2h}} \cdot \prod_{i \in I^c} [v_i]_{\chi_0} \quad (12)$$

Splitting the sum over  $I$  according to order and recalling from §2.3, (3) that  $\sum_{r=1}^k (-1)^{r-1} N_{k-s, r-1} = (-1)^{k-s}$ , the expression (12) reduces to

$$\sum_{s=0}^k (-1)^{k-s} \sum_{I \subseteq [k], |I|=s} \prod_{i \in I} [v_i]_{\chi_{-2h}} \cdot \prod_{i \in I^c} [v_i]_{\chi_0} = \prod_{i=1}^k ([v_i]_{\chi_{-2h}} - [v_i]_{\chi_0})$$

Thus, its (partial) derivative as used above, is a product of  $(k-1)$  differences of the form  $[v_i]_{\chi_{-2h}} - [v_i]_{\chi_0}$ ,  $i \neq k$ , and consequently  $[\sigma_k(v_1 \wedge \cdots \wedge v_k)]_{\bar{\chi}_0 dx}$  can be written as a sum of terms, each of which is a product of  $k$  factors,  $k-1$  of which being differences of this form; hence it is divisible by  $h^{k-1}$ .

The proof that  $[\sigma_k(v_1 \wedge \cdots \wedge v_k)]_{\bar{\chi}_{2h} dx}$  is divisible by  $h^{k-1}$  is identical to the previous proof, with  $\chi_0, \chi_{-2h}$  replaced by  $\chi_h, \chi_{3h}$ .  $\square$

**Lemma 7.8.** *Extending  $\bar{\phantom{x}}$  from the cochains  $B$  on the dimension one lattice to the cochains  $A$  on the dimension  $n$  lattice by tensor product, the cummulant  $\sigma_k : S^k A \rightarrow \bar{A}$  of the map  $\bar{\phantom{x}}$  so defined is divisible by  $h^{k-1}$  for  $k > 1$ .*

*Proof.* By Lemma 7.6, the tensor product of any two structures for which the  $k^{\text{th}}$  cumulants are divisible by  $h^{k-1}$  will also satisfy this property, since the term

in the sum in Lemma 7.6 indexed by the partition  $I_1 \cup \dots \cup I_r$  will have  $h$ -order,

$$\sum_{j=1}^r (|I_j| - 1) + (r - 1) = k - 1$$

Combining with Lemma 7.7, the result follows.  $\square$

*Remark 7.9.* The conclusion is that for adjacent scales, the integration map  $\bar{\phantom{x}}$  between them  $A \rightarrow \bar{A}$ , provides an induced map (given by composition with cumulant bijections and whose Taylor coefficients are identified by Lemma 7.3 with commutative cumulants)  $\sigma : S^*A \rightarrow S^*\bar{A}$  which satisfies  $\sigma \circ D = \bar{D} \circ \sigma$  as maps (coderivations)  $S^*A \rightarrow S^*\bar{A}$ . The equality of Taylor coefficients of the maps on the two sides of this equation gives the exact relation between the systems of  $k$ -brackets (infinitesimal cumulants) at the coarse and fine scales, namely a collection of relations,

$$\begin{aligned} [\sigma_1, \delta'_1] &= 0 \\ [\sigma_1, \delta'_2] + [\sigma_2, \delta'_1] &= 0 \\ &\dots \\ [\sigma_1, \delta'_k] + [\sigma_2, \delta'_{k-1}] + \dots + [\sigma_k, \delta'_1] &= 0 \end{aligned}$$

where the  $k^{\text{th}}$  equation is an equality between maps  $S^kV \rightarrow \bar{V}$  ( $\sigma$ 's extended as coalgebra maps and  $\delta$ 's extended as coderivations). Here an extended notion of commutators is being used whereby for a map  $b : S^*V \rightarrow S^*\bar{V}$  we denote by  $[b, x]$  the difference  $b \circ x - \bar{x} \circ b$ , which is a map  $S^*V \rightarrow S^*\bar{V}$ , for any symbol  $x$  which represents a map  $S^*V \rightarrow S^*V$  and for which there is an analogous map on the 'bar'-side  $\bar{x} : S^*\bar{V} \rightarrow S^*\bar{V}$ .

The first equation is just the statement that  $\bar{\phantom{x}}$  is a cochain map. The second relation in long-hand states that

$$\overline{[v, w]} - [\bar{v}, \bar{w}] = \bar{\delta}' \circ \sigma_2(v \wedge w) - \sigma_2(\delta'v \wedge w + (-1)^{|v|}v \wedge \delta'w)$$

The above collection of relations between the  $k$ -brackets  $\delta'_k$  and  $\bar{\delta}'_k$  (which are of order  $h^{k-1}$  for  $k > 1$ , by Theorem 4.1) defines the collection of maps  $\sigma_k : S^kA \rightarrow \bar{A}$  (of order  $h^{k-1}$  for  $k > 1$  by Lemma 7.8) to be a morphism (in the sense of Definition 6.2) between the two binary QFT algebras  $A$  and  $\bar{A}$  (of Theorem 6.4). Hence we obtain a formulation of Theorem 1.3(B) as follows.

**Theorem 7.10.** *Considering the binary QFT algebra  $(A(h), \wedge, \delta')$  of Theorem 6.4 for scales  $h = h_0 \cdot 2^{-m}$ ,  $m \in \mathbb{N}$ , gives rise to an inverse system of binary QFT algebras related by binary QFT morphisms between the structures at the various scales.*

The chain map  $\iota$  of §7.2 from the coarse chain complex  $(\bar{C}, \bar{\partial})$  to the fine chain complex  $(C, \partial)$ , will according to §7.3 induce maps  $S^*\bar{C} \rightarrow S^*C$  which intertwine the infinitesimal cumulants at the coarse and fine levels of  $(C, \partial)$  of

Theorem 5.2. The following theorem is deduced by exactly similar arguments to those in the proof of Theorem 7.10, based on direct computation of the cumulants of the explicit chain map using Lemmas 7.3 and 7.6.

**Theorem 7.11.** *Considering the binary QFT algebra  $(C(h), \wedge, \partial)$  of Theorem 6.3 for scales  $h = h_0 \cdot 2^{-m}$ ,  $m \in \mathbb{N}$ , gives rise to a direct system of binary QFT algebras related by binary QFT morphisms between the structures at the various scales.*

## Background Appendix

At the continuum level there are two product structures on distinct but isomorphic spaces which play different roles. The one on differential forms is familiar in topology because of integration and mapping properties. The other on multivector fields is familiar in differential geometry. There is a third contraction product between a  $k$ -differential form and a  $j$ -multivector field giving a  $(j - k)$ -multivector field if  $j > k$ , a  $(k - j)$ -differential form if  $k > j$  and a function if  $j = k$  which by definition is both a differential form and a multivector field.

The  $j > k$  case of the contraction product has an interpretation in algebraic topology, *e.g.* Whitney's cap product, while the  $k > j$  case is used often in differential geometry, *e.g.* in Cartan's magic formula.

### One algebra structure

The integration of a  $k$ -differential form over an oriented  $k$ -surface defines a  $k$ -cochain, *i.e.* a linear function on  $k$ -chains, the term used for linear combinations of oriented  $k$ -surfaces. There is the geometric boundary map on chains called  $\partial$ , taking a  $(k + 1)$ -chain to a  $k$ -chain, and thus a dual operator on cochains referred to here as  $\delta$ , taking a  $k$ -cochain to a  $(k + 1)$ -cochain. Poincaré calculated and defined the general exterior  $d$  on differential forms and showed  $dd = 0$ . Stokes' theorem says  $\int : (k - \text{forms}) \rightarrow (k - \text{cochains})$  intertwines Poincaré's exterior  $d$  acting on forms or integrands with  $\delta$  acting on cochains. This was Poincaré's picture *c.*1900 and marked the birth of algebraic topology.

Using exterior or Grassmann algebra, differential forms have a fully defined graded commutative and associative algebra structure and a differential  $d$  of square zero which is a derivation of this product. It is of great advantage that this differential algebraic structure on forms is natural for all appropriate mappings between manifolds. See [3], [6] and [9].

One knows in topology that it is impossible to break space into finite cells and to construct a finite-dimensional algebraic structure on the cochains that reflects perfectly the properties of  $d$  and this product on differential forms. This possibility in the case of forms forces the infinite-dimensionality of forms.

### The other algebra structure

Secondly, at the continuum level, there is the exterior product structure on the multivector fields, by definition the cross sections of the exterior powers of the tangent bundle. This product is similar in appearance to the graded algebra structure of differential forms. But it has a quite different meaning.

Given a volume measure on the ambient manifold,  $k$ -multivector fields via contraction with  $k$ -differential forms followed by integration of the resulting function, define elements in the dual space of  $k$ -differential forms. There is thus defined a dual operator of degree  $-1$ , call it  $\partial$ , defined by linear duality from the exterior derivative  $d$  on forms. This  $\partial$  is consistent with the geometric boundary operator  $\partial$  on chains mentioned above. Thus the multivector fields have a graded commutative associative product and a square zero operator of degree  $-1$ , in the presence of a volume measure. This structure is natural for diffeomorphisms that preserve volume.

For a manifold of dimension  $d$ , contracting multivector fields with a fixed volume form yields an isomorphism between multivector fields and differential forms which relates  $\partial$  to exterior  $d$ , providing a geometric version of Poincaré duality.

There is a difference between product rules for the two structures on differential forms and on multi-vector currents that arises because  $d$  is a derivation of its product, the exterior product of forms, while  $\partial$  is not a first order derivation.

A bracket operation on multivector fields is defined as the deviation of  $\partial$  from being a derivation of the exterior product of multi-vector fields. By construction,  $\partial$  is a derivation of this bracket. It turns out, again remarkably, that this deviation bracket is itself a derivation of the exterior product in one of its arguments with the other held fixed. This is called the *second order derivation property* of the continuum  $\partial$  relative to its product. It is noteworthy that this second order property plus  $\partial^2 = 0$ , implies the Jacobi identity holds for this bracket of multi-vector fields.

### Metric or symplectic structures

If the volume form is that of a Riemann metric which is used to identify multivector fields and forms of the same degree and thus also their respective products, then  $\partial$  becomes the metric adjoint of exterior  $d$ . Recall that this adjoint is the conjugate of exterior  $d$  by the Hodge star operator. It is often denoted  $d^*$  or  $\partial$ , has degree  $-1$  and its square is zero. One sees a significant universal identity between  $d$ , the exterior product and the Hodge star operator on forms for all Riemannian manifolds: “the adjoint of exterior  $d$  is a second order derivation of the exterior product of forms”.

If the identification between tangent space and cotangent space is defined by a closed nondegenerate two form there is an analogous second order operator of square zero leading to the the physicist’s Batalin-Vilkovisky calculations motivating the definition in [2] of a binary QFT algebra. The latter is realized at

the discrete level above while the former requires the continuum.

**Funding:** This work was supported by the US-Israel Binational Science Foundation [grant 2016219 to R.L.&D.S.]; Balzan Foundation [2014 Prize to D.S].

### **Postscript: Sir Michael Atiyah**

In August 1966 the third author, age 25, on the road between Warwick and Southampton to board a ship home, was graciously invited to lunch by Professor Michael Atiyah at his college in Oxford to discuss math questions related to  $K$ -theory orientations for PL manifolds, posed the evening before on the telephone. On the way in to dining, some college fellows were congratulating Professor Atiyah on his just awarded Fields Medal received in Moscow.

Professor Atiyah offered advice during a jolly lunch about good, better and best fields of Mathematics. It was amazing how he could make conceptual connections between different fields. Still, young people could resist the advice of their elders: PL manifolds, good; smooth manifolds-differential geometry, better; analysis-algebraic geometry-number theory, best.

At the 1968 Global Analysis Summer Conference in Berkeley, Michael Atiyah described, during a swim in the Strawberry Canyon pool, “triviality and the six sphere” plus a fascinating way to understand, via complex conjugation, the holomorphic structure on the real Grassmanian of oriented two-planes in  $\mathbb{R}^n$ .

In 1969, Michael Atiyah, at MIT’s cafeteria related Hironaka’s resolution of singularities to Hormander’s characteristic variety for hyperbolic PDE’s.

During Professor Atiyah’s tenure *c.*1970 at the IAS in Princeton, this writer showed up at his office door very early one morning with a zany idea. Atiyah managed to relate Adams operations and  $K$ -theory of PL manifolds to Brumer’s 1967 linear independence over  $\mathbb{Q}$  of  $p$ -adic logarithms of algebraic numbers.

Michael Atiyah was also telling in those days of the early 70’s, of a very neat way to think about proving groups of odd order were solvable. This, by finding a fixed point theorem for actions of such groups on complex projective spaces. His very recent attempts on the Feit-Thompson theorem related to this idea.

Two years ago there were meetings at breakfast several mornings in Shanghai at the Alain Connes fest to critically discuss his reasoning why  $S^6$  should not have an integrable complex structure. Sometimes, he left the table grumbling, but returned to battle the next day. The discussions did not resolve the issue.

With Michael Atiyah’s strong intuition and deep insight across many fields, it seemed important then, and it seems important now, to leave no stone unturned in pursuing his ideas. We will miss Sir Michael Atiyah’s unfettered enthusiasm for deep and broad mathematics and his exciting ideas for its advance.

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