

# HOMOTOPY THEORY OF THE MASTER EQUATION PACKAGE APPLIED TO ALGEBRA AND GEOMETRY: A SKETCH OF TWO INTERLOCKING PROGRAMS

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ABSTRACT. We interpret mathematically the pair (master equation, solution of master equation) up to equivalence, as the pair (a presentation of a free triangular dga  $T$  over a combination operad  $O$ , dga map of  $T$  into  $C$ , a dga over  $O$ ) up to homotopy equivalence of dgOa maps, see Definition 1 below.

We sketch two general applications:

I to the theory of the definition and homotopy theory of infinity versions of general algebraic structures including noncompact frobenius algebras and Lie bialgebras. Here the target  $C$  would be the total  $Hom$  complex between various tensor products of another chain complex  $B$ ,  $C = Hom B$ ,  $O$  describes combinations of operations like composition and tensor product sufficient to describe the algebraic structure and one says that  $B$  has the algebraic structure in question.

II to geometric systems of moduli spaces up to deformation like the moduli of  $J$  holomorphic curves. Here  $C$  is some geometric chain complex containing the fundamental classes of the moduli spaces of the geometric problem.

We also discuss analogues of homotopy groups and Postnikov systems for maps and impediments to using them related to linear terms in the master equation called anomalies.

## INTRODUCTION AND SKETCH

Certain combination operads arise in the study of algebraic structures and moduli spaces. For any operad  $O$  one may define differential graded algebras over  $O$ . Let us call them dgOa's. Fixing  $O$  they form an obvious category where the maps are dgOa maps. We will make use of a derived homotopy category based on free resolutions of dgOas and a notion of homotopy between dgOa maps. Resolutions give a procedure to replace any dgOa by a nilpotent analogue of a free dgOa. There are two similar classes of examples relevant here where the combination operad  $O$  describes compositions or tensor products of multilinear operations in application I and where the combination operad  $O$  describes gluing or union of geometric chains in application II.

### 1. THE SETUP OF APPLICATION I

Consider collections of  $j$  to  $k$  operations for various  $j$  and  $k$  positive that define examples of algebraic structures like (noncompact) frobenius algebra or Lie bialgebra. Such examples of algebraic structures, because multiple outputs appear, cannot themselves be described as algebras over operads but rather as algebras over dioperads, properads or props. Dioperads correspond to inserting only one of the multiple outputs into the inputs. Properads correspond to

allowing multiple outputs to be inserted into the inputs. Props include as well the tensor product of operations. We won't use these concepts per se because any prop, properad or dioperad can be described as an algebra over the combination operad describing the combinations of operations required in the corresponding definition.

There are a few choices for which combinations of operations are allowed.

For example a Lie bialgebra is defined by a two-to-one product and a one-to-two coproduct. These are both skew commutative and satisfy three quadratic relations requiring one variable substitution. There are four combination operads that could be employed here. 1) all the operations freely generated by composing these two in all possible ways with only one output inserted in to an output; 2) same as 1) but with multiple outputs inserted into multiple outputs; 3) same as is 1) with tensor products thrown in; 4) same as 2) with tensor products thrown in.

Since the generating operations are both graded skew symmetric the combination operads 1) 2) 3) 4) can be described as follows: 1) by gluing trivalent trees with labeled inputs and outputs; 2) by gluing all connected directed trivalent graphs with labeled non empty sets of input and output vertices and no directed cycles; 3) as in 1) but not necessarily connected; 4) same as in 2) but not necessarily connected. On the other hand a noncommutative Frobenius algebra (non compact version) is defined by an associative product and a coassociative coproduct satisfying two quadratic compatibility relations. The four corresponding combination operads will be described by the above graphs where the half edges at a vertex have the additional structure of a cyclic order. Infinity versions of these algebraic structures will use higher valence graphs as well.

## 2. THE SETUP OF APPLICATION II

Consider gluing operations describing compactifications in a system of moduli spaces coming from some geometric problem involving, for example, riemann surfaces, connections on  $G$  bundles or configuration spaces of manifolds. The main point is a hereditary property of compactifications of these moduli spaces. The homotopy theory of the master equation described here becomes relevant if the points added in the compactifications of these moduli spaces can be described in terms of other moduli spaces of the same system. This description uses gluing operations on moduli spaces or their fundamental chains and these operations in turn are described abstractly by a relevant gluing or combination operad.

The category of dgOas will be used in the two settings above via its associated homotopy theory.

## 3. BASIC FACTS OF DGOA HOMOTOPY THEORY

1) One knows that there are free dgOa algebras associated to any system of generating vector spaces with zero differential. For example in the associative case the free algebra is the tensor algebra on the generating spaces without the unit or ground ring term, namely the augmentation ideal of the tensor algebra with unit.

One says a dgOa is free if it is free in this sense after suppressing the differential. A free dgOa is called triangular if there is a partial ordering on the generators, with all descending chains

finite, so that the differential of any generator is a sum of  $O$ -operations applied to strictly smaller generators for the partial order.

This is the analog of nilpotent space or nilpotent differential Lie algebra in usual homotopy theory.

**Lemma 1.** *If  $A$  is any dgOa there is a map  $T \rightarrow A$  from a free triangular dgOa  $T$  to  $A$  inducing an isomorphism on homology. Such maps are called resolutions.*

This follows from a straightforward induction.

The first step of one induction, which is not the most efficient, is to choose a generating set for the homology of  $A$ , form the free algebra on these over  $O$ , define the differential to be zero there and map the generators to cycles representing the named homology classes. The second step of this induction is to add generators to the domain whose differentials put in a spanning set of homology relations among the cycles in  $A$  chosen in the first step. These second stage generators are mapped to elements in  $A$  which exist because these homology relations are satisfied. The third step puts in relations that kill the kernel of this dgOa map etc. There are many constructions of resolutions.

In the homotopy theory of topological spaces the analog of resolution in this sense is the Quillen plus construction. It is unique up to homotopy.

2) One also knows how to regain uniqueness of resolutions in the case of homological algebra using chain equivalences and chain homotopies. Chain homotopies in a partially ordered context can be defined inductively by solving linear equations like  $dx = y$ , with  $y$  determined inductively and  $x$  unknown. There is an analogous but more nontrivial notion of homotopy between dgOa maps from a free triangular dgOa  $T$  into  $A$  an arbitrary dgOa. The theory follows the same line as developed in [8] for dga's over the graded commutative operad. Now one is inductively solving a triangular system of equations  $dx = \text{sum of } O\text{-operations of } y\text{'s}$  with  $x$  unknown and the RHS determined inductively. This theory of homotopies is described in detail for associative algebras over novikov rings in [1]. This notion of homotopy is also used in papers by Markl [5, 6, 7].

Using natural obstruction theory arguments one can show two lemmas:

**Lemma 2.** *A dgOa map from  $T$ , a free triangular dgOa, into any dgOa  $A$  can be lifted up to homotopy into  $B$  for any dgOa map  $B \rightarrow A$  which induces an isomorphism on homology. The lift is unique up to homotopy.*

**Corollary 1.** *A map between free triangular dgOa's  $T \rightarrow T'$  inducing isomorphisms on homology is a homotopy equivalence in the usual sense: there is a map  $T' \rightarrow T$  so that each composition is homotopic to the identity.*

**Lemma 3.** *Given two resolutions  $T \rightarrow A$  and  $T' \rightarrow A$  there is a homotopy equivalence between  $T$  and  $T'$  which is unique up to homotopy such that the diagram into  $A$  commutes up to homotopy.*

**Definition 1.** *(homotopy equivalence of maps) Two maps  $T \rightarrow A$  and  $T' \rightarrow A'$  are said to be homotopy equivalent if there are homotopy equivalences  $f$  between  $T$  and  $T'$  and  $g$  between the resolutions of  $A$  and  $A'$  so that the lifted maps from  $T$  and  $T'$  to these resolutions together with  $f$  and  $g$  form a commutative square up to homotopy.*

#### 4. APPLICATION I: GENERAL ALGEBRAIC STRUCTURES, INFINITY VERSIONS THEREOF AND THEIR HOMOTOPY THEORY

Any algebraic structure described by  $j$  to  $k$  operations, for various  $k$  and  $j$  positive, on a chain complex  $C$  can be viewed as a dgOa map for some composition operad  $O$ . The domain of this “structure map” is a dgOa whose presentation in terms of generators and relations defines the algebraic structure in question where the combination operad  $O$  is determined by the kind of combinations required to express the relations. Actually  $O$  may then be enlarged if desired by adding further combinations as illustrated in the examples above. The range of the structure map is the total *hom* complex, denoted  $Hom(C)$  between the various tensor powers of  $C$  endowed with the composition and tensor product operations labeled by the operad  $O$ .

**Definition 2.** (*algebraic structures and infinity versions of algebraic structures*) *An algebraic structure on  $B$  is defined to be a dgOa map of any dgOa into  $HomB$  regarded as a dgOa where  $O$  is the combination operad describing the operations of composition and tensor product considered as part of the structure. If the domain of the dgOa map is a free triangular dgOa the structure is called an infinity algebraic structure. Any algebraic structure has an infinity version obtained by replacing the domain by a resolution of the domain. Forming composition with the resolution map associates with one particular instance of an algebraic structure a particular instance of an infinity algebraic structure. One may think of any infinity algebraic structure as the infinity version of its own homotopy type.*

**Definition 3.** *If  $D$  is another chain complex a “HomO quasi isomorphism” from  $C$  to  $D$  is a homotopy class of homotopy equivalences between a resolution of  $HomC$  and a resolution of  $HomD$  as dgOa algebras.*

It follows from the definitions that one may transport infinity algebraic structures up to equivalence back and forth between  $C$  and  $D$  by a HomO quasi isomorphism.

To use this notion the following lemma is useful.

**Lemma 4.** *Suppose  $C$  and  $D$  are two quasi isomorphic chain complexes over the rationals. Then the dgOa algebras  $HomC$  and  $HomD$  have homotopy equivalent resolutions. In other words, an ordinary quasi isomorphism implies a HomO quasi isomorphism.*

The idea of the proof is to prove it for the case when  $D = H$  is the homology of  $C$ . In this case there is a dgOa map from  $HomH$  to  $HomC$  using a purely algebraic analogue of a hodge decomposition of  $C$ . A multilinear operation on tensor products of harmonic elements can be extended to all of  $C$  by defining it to be zero on tensors with exact or coexact factors. This map induces an isomorphism on homology of the *Hom* complexes.

**Definition 4.** *Two algebraic structures with possibly different presentations on possibly different chain complexes are called quasi isomorphic or homotopy equivalent if their associated infinity versions have homotopy equivalent structure maps (Definition 1) after lifting them to resolutions of the Hom complexes.*

#### 5. APPLICATION II: NON LINEAR HOMOLOGY OF SYSTEMS OF GEOMETRIC MODULI SPACES

1) Various geometric problems that resonate with quantum or string discussions in theoretical physics give rise to systems of oriented pseudomanifolds with boundary where the codimension

one pieces of the boundary can be described by gluing and intersection operations applied to earlier pseudomanifolds in the system. A correct formal description of the pieces in such a theory leads to a free triangular dgOa where the combination operad  $O$  describes the operations used in this description. Let  $X$  denote the tuple labeling all the formal moduli pieces of the system. Then this description usually takes the form of a “master equation” like  $dX + X * X = 0$  or  $dX + LX + X * X = 0$ . Here  $*$  denotes the binary gluing operations of the description and  $L$  the unary operations required in the description. The formal identity  $dd = 0$  follows from the geometric fact that the boundary of a boundary of the formal moduli is zero. We obtain from the master equation at this formal level a presentation of a free triangular dgOa. The partial ordering may come from a dimension consideration or from an energy consideration.

Solving the PDEs defining the formal moduli yields a set of chains solving the master equation. The solutions of these equations in the geometric chain complex  $C$  defines a dgOa map from  $T$  the triangular free dgOa into  $C$  the geometric dgOa.

So we see the same dgOa formalism that applies to algebraic structures also applies to give a description of systems of moduli spaces assuming the hereditary property: in the compactifications the ideal points are described by lower (in the sense of dimension or energy) moduli spaces of the system. The difference is that now the range of the dgOa map is not necessarily homotopy equivalent to a complex of the form  $C = Hom B$  as it was in the case of algebraic structures.

- 2) Varying the choices in the geometric equations perturbing the PDEs, e.g., to create transversality, is meant to lead to a homotopy equivalence of the dgOa map associated to this moduli package.
- 3) The linear terms in the master equation as just described are called anomalies. They make the above discussion vulnerable to being homotopically trivial. This is analyzed by looking at the linearized homology of the free triangular dgOa which will be discussed next.

## 6. POSTNIKOV SYSTEMS AND MINIMAL MODELS IN THE DGOA CONTEXT

1) Given a free triangular dgOa  $T$  we can of course form the usual or global homology which is an algebra over  $O$ . We can also form a linearized chain complex and its homology which is called the linearized homology. The linearized complex is the quotient of the free dgOa by the  $d$  submodule defined by the image of  $O$  operations with at least two inputs. The linearized homology behaves like the homotopy groups of  $T$  or rather their dual spaces. The natural map from the global homology of  $T$  to the linearized homology of  $T$  is analogous to the dual of the Hurewicz homomorphism in topology from homotopy to homology.

When  $O$  is the graded commutative operad the dual of linearized homology of a dgOa has the structure of a Lie algebra which is the leading part of an  $L$  infnty structure. In topology this corresponds to Whitehead products and higher order Whitehead products on homotopy.

This generalizes in the following way over the rationals.

- 2) Let  $H$  denote the linearized homology of a free triangular dgOa  $T$ .

**Corollary 2.** *There is a triangular differential in  $O(H)$ , the free dgOa generated by  $H$ , so that this dgOa is homotopy equivalent to  $T$ .*

The proof in this setting is direct. View the differential in  $T$  as a dgOa map from a fixed free triangular dgOa into  $Hom$  of the linearized complex. Lift this to  $HomH$  using lemma 2 and the proof of lemma 4.

Note that the differential in  $O(H)$  consists of maps from  $H$  to various tensor products of  $H$  with itself. This is a set of coproducts satisfying quadratic identities. In the dual picture these provide the indicated generalization of Whitehead products and higher order products to the  $O$  context. The commutative case of Corollary 2 appeared in [8] but the  $L$  infinity interpretation was missing before the work of Hinich and Schechtman [2, 3]. Corollary 2 in this generality is certainly due to Markl [5, 6, 7]. See also Kadeishvili [4] and the thesis of Bruno Vallette [11].

3) Minimal models.

**Definition 5.** *The  $O(H)$  version of the homotopy type of  $T$  just described is called the minimal model of the homotopy type.*

The minimal model is built up inductively by adding layers of dual homotopy groups in an algebraic way that models combining base and fibre to get the total space in a fibration. This is the analogue in the free triangular dgOa world of the nilpotent or untwisted Postnikov system in homotopy theory.

4) More generally there is the algebraic analogue of the Postnikov system of a map in homotopy theory where the stages follow the homotopy groups of the fibre of the map. Namely given  $T \rightarrow A$  one inductively adds generators to  $T$  and extends the map to make it into a homology isomorphism i.e., a resolution of  $A$ . There are minimal versions of this process which reveal a set of invariants of the homotopy type of the original map.

## 7. ANOMALIES

If starting from the geometry one finds a master equation of the form  $dX + X * X +$  higher order terms... = 0, then the domain of the dgOa map is in minimal form and each component of  $X$  represents a nontrivial homotopy group. Thus there are in general nontrivial invariants of such dgOa maps up to homotopy equivalence. On the other hand if there are linear terms in the master equation  $dX + LX + X * X +$  higher terms... = 0, the linearized differential is the unary operator  $L$  and its homology might be zero. In various geometric contexts it is sometimes possible to use symmetry or other geometric devices to kill some of the linear terms of  $L$  by additional gluing or filling in. Reducing  $L$  increases the linearized homology and thus the fund of possible invariants. See [1, 9, 10, 12].

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