# String Topology

Moira Chas and Dennis Sullivan

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### Abstract

Consider two families of closed oriented curves in a manifold  $M^d$ . At each point of intersection of a curve of one family with a curve of the other family, form a new closed curve by going around the first curve and then going around the second. Typically, an *i*-dimensional family and a *j*-dimensional family will produce an i + j - d + 2-dimensional family. Our purpose is to describe a mathematical structure behind such interactions.

## 1 Introduction

By the *string homology* of a manifold  $M^d$  we mean the equivariant homology of the continuous mapping space  $Map(\mathbb{S}^1, M^d)$  with the circle symmetry of rotating the domain. The goal of this paper is to expose the following theorem and its underpinnings:

**Theorem 6.1** On the string homology of a smooth or combinatorial oriented manifold  $M^d$  there is a natural graded Lie algebra structure of degree (2-d).

This structure called the *string bracket* comes from the interaction of closed oriented curves in  $M^d$ . At each point of intersection of two such curves, one can form the oriented curve obtained by going around one and then around the other. Transversality will be used on the cells computing string homology to define the string bracket of the theorem. For an *i* cell and a *j* cell of strings there will typically be a (i+1)+(j+1)-d = (i+j)+(2-d) dimensional set of intersection points. If we picture a bracket by two circles touching, then a triple bracket means a third circle touches the result along one arc or the other. Thus, the Jacobi identity can be viewed as in Figure 1, where the two terms on left appear on the right with two other terms that are geometrically identical up to sign. This Lie algebra is quite non-trivial



Figure 1: Explanation of Jacobi identity

for surfaces of genus larger than one (see reference to Goldman and Wolpert below).

In the process of analyzing the above argument we found a more basic structure called the *loop product* on the *loop homology*, the ordinary homology of the free loop space  $\operatorname{Map}(\mathbb{S}^1, M^d)$ . A version of the loop product • of two cells in  $\operatorname{Map}(\mathbb{S}^1, M)$  is defined for each point z on  $\mathbb{S}^1$  by first intersecting in  $M^d$  the two cells obtained by evaluating at this point z and then composing the loops at these intersection points.

There are homotopies making the loop product on loop homology into an associative graded commutative algebra. As in Gerstenhaber's basic paper [1], there is a preferred homotopy for the graded commutativity (denoted \*) which leads by symmetrization to a second operation  $\{,\}$  on loop homology. This operation, called *loop bracket* satisfies the Jacobi identity where now it is convenient to grade  $\mathbb{H}_*$  by subtracting d from the usual geometric grading so that  $\bullet$  and  $\{,\}$  become operations of degree zero and one respectively,

$$\mathbb{H}_i \otimes \mathbb{H}_j \xrightarrow{\bullet} \mathbb{H}_{i+j},$$
$$\mathbb{H}_i \otimes \mathbb{H}_j \xrightarrow{\{,\}} \mathbb{H}_{i+j+1},$$

instead of -d and (-d+1) respectively.

We arrive at the following result.

**Theorem 4.7** The loop product • with the loop bracket  $\{,\}$  makes the loop homology  $\mathbb{H}_{\star}$  (the ordinary homology of the free loop space) into a Gerstenhaber algebra, namely:

- (1) The loop product  $\bullet$  defines a graded commutative, associative algebra.
- (2)  $\{,\}$  is a Lie bracket of degree 1, which means that for each  $a, b, c \in \mathbb{H}_*$

$$(i) \ \{a,b\} = -(-1)^{(|a|+1)(|b|+1)} \{b,a\}$$
  

$$(ii) \ \{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+1)(|b|+1)} \{b,\{a,c\}\}$$
  

$$(3) \ \{a,b \bullet c\} = \{a,b\} \bullet c + (-1)^{|b|(|a|-1)} b \bullet \{a,c\}.$$

Now consider the circular symmetry of the loop space  $Map(\mathbb{S}^1, M) = \mathbb{L}(M)$ , and the associated degree +1 operator  $\Delta$  on homology

i cycle in  $\mathbb{L}(M) \longrightarrow i + 1$  cycle in  $\mathbb{L}(M) = i$  cycle  $\times \mathbb{S}^1 \xrightarrow{action} \mathbb{L}(M)$ .

It turns out that  $\Delta$  is a second order operator in the sense of commutative algebra, i.e., the binary operation, the deviation of  $\Delta$  from being a derivation of  $\bullet$ , is a derivation in each variable. One also has easily that  $\Delta \circ \Delta = 0$ . We come to

**Theorem 5.4** The loop product • and the operator  $\Delta$  make the loop homology (the ordinary homology of Map( $\mathbb{S}^1, M$ )) into a Batalin Vilkovisky algebra, namely:

- (1) is a graded commutative associative algebra.
- (2)  $\Delta \circ \Delta = 0.$
- (3)  $(-1)^{|a|}\Delta(a \bullet b) (-1)^{|a|}\Delta a \bullet b a \bullet \Delta b$  is a derivation of each variable.

The connection between these two results is that conditions (1), (2), and (3) of Theorem 5.4 imply formally that the binary operation defined by the deviation satisfies graded Jacobi so that a Batalin Vilkovisky algebra is a special type of Gerstenhaber algebra. We prove Theorem 5.4 by constructing a chain homotopy between the  $\{x, y\}$  of Theorem 4.7 and  $(-1)^{|x|}\Delta(x \bullet y) (-1)^{|x|}\Delta(x) \bullet y - x \bullet \Delta(y)$ . We note that in the discussion of these two theorems there are two independent proofs of Jacobi for the loop bracket.

Finally we come again to the string bracket on string homology, the equivariant homology of the mapping space of  $\mathbb{S}^1$  into M. The approach referred to above is to use intersection theory on chains to define the string bracket and verify the Jacobi identity directly on the level of transversal triples of chains.

The second approach to the string bracket uses the loop product  $\bullet$  discussed above. Consider the degree +1 operation *lift* from equivariant chains to ordinary chains corresponding to replacing an *i*-chain in the base of an S<sup>1</sup> fibration by the *i*+1 chain which is the preimage in the total space. Consider also the operation *project* which simply projects chains in the total space to the base. Then we define the string bracket in terms of the loop product by the formula

$$[x, y] = \text{project}(\text{lift } x \bullet \text{ lift } y).$$

(In our calculations, Section 6 we denote *lift* by M, because lift means mark a point in all possible ways on a closed curve without a mark (a string). We denote *project* by E because project means erase the marked point in a marked curve (loop) to get an unmarked curve (string)).

The composition (*lift*)• (*project*) induces the operator  $\Delta$  above on loop homology, and the Jacobi identity for [,] follows by direct calculation from the properties of the Gerstenhaber qua Batalin Vilkovisky algebra structures above.

We arrive at Theorem 6.1 stating that the string bracket defines a graded Lie algebra structure on string homology. The geometric degree is (2-d) for the usual grading of string or equivariant homology.

The same argument constructing the binary operation  $[,] = \bar{m}_2$ , the string bracket, constructs ternary, etc. operators  $\bar{m}_3, \bar{m}_4, \ldots$ 

Now extend each  $\overline{m}_k$  to a coderivation  $m_k$  to  $\Lambda \mathcal{H}_{\star}$  (see Section 6).

One knows the Jacobi identity for the string bracket  $\bar{m}_2$  is equivalent to the relation  $m_2 \circ m_2 = 0$  for the associated coderivation  $m_2$ .

The Jacobi relation for  $\bar{m}_2$  generalizes to the entire collection  $\{\bar{m}_2, \bar{m}_3, \ldots\}$  in the following way.

**Theorem 6.2** The associated coderivations  $\{m_2, m_3, \ldots\}$  of the free commutative coalgebra  $\Lambda \mathcal{H}_{\star}$  on the string homology  $\mathcal{H}_{\star}$  satisfy:

- (i)  $m_k \circ m_k = 0$ , for  $k = 2, 3, 4 \dots$
- (*ii*)  $m_k \circ m_r + m_r \circ m_k = 0$  for  $k, r = 2, 3, 4, \dots$

These coderivations  $m_k$  combine in various ways to define coderivations of square zero on  $\Lambda \mathcal{H}_{\star}$ . Such a differential is one definition of a Lie<sub> $\infty$ </sub> or strong homotopy Lie algebra structure on  $\mathcal{H}_{\star}$ . Thus we have

**Corollary 6.3** There exists an uncountable family  $\{\delta_{\Lambda}\}$  of  $\text{Lie}_{\infty}$  structures on the string homology. Namely, for each  $\Lambda \subset \{2, 3, \ldots\}$ ,

$$\delta_{\Lambda} \colon \Lambda \mathcal{H}_{\star} \longrightarrow \Lambda \mathcal{H}_{\star} \text{ defined as } \delta_{\Lambda} = \sum_{\lambda \in \Lambda} m_{\lambda}$$

is a coderivation which satisfies  $\delta_{\Lambda} \circ \delta_{\Lambda} = 0$ .

If we examine the string bracket when d = 2 we find a Lie bracket structure on the vector space of components of the space of closed curves in a surface. For surfaces of genus larger than zero this is the non-trivial bracket discovered in the 80's by Wolpert [12] and Goldman [3]. That discovery was strongly related to the symplectic structure on Techmuller space [12] and the symplectic structure of other spaces of flat connections over a surface [3]. (For more discussion see Section 7). When the genus is zero, i.e., M is a two sphere, the loop product becomes non-trivial in the higher dimensional algebraic topology of the free loop space of  $\mathbb{S}^2$ .

The S<sup>2</sup>-calculation is part of a general structure based on the diagram relating the intersection product on ordinary homology  $(H_{\star}(M), \wedge)$  with the loop product on loop homology  $(\mathbb{H}_{\star}, \bullet)$  and the Pontryagin product on the based loop space homology  $(H_{\star}(\Omega), \times)$ ,

(1) 
$$(H_{\star}(M), \wedge) \xrightarrow{\text{constant loops}} (\mathbb{H}_{\star}, \bullet) \xrightarrow{\text{intersection with a fiber}} (H_{\star}(\Omega), \times)$$

Both maps are ring homomorphisms. The first is an injection onto a direct summand showing the loop product is an extension of the classical intersection product. The image of the second map is a graded commutative subalgebra. For  $\mathbb{S}^2$ ,  $(H_{\star}(\Omega), \times)$  is the tensor algebra on one generator  $\eta$  in degree one and the image is the subalgebra generated by  $\eta^2$ . This shows the loop product is non-trivial for  $\mathbb{S}^2$  (Section 9). For more complete calculations we can augment the diagram (1) with a relation between the usual cap product operation,  $\cap$ , and the loop product. This is described by

**Theorem 8.2** For each x, y homology classes and compatible pair of classes (A, a)

loop product 
$$(A \cap (x \otimes y)) = a \cap (x \bullet y)$$
.

(see Section 8 for the definitions and the proof, and Section 9 for relevant algorithms.)

The two approaches here to the string bracket, direct geometry and via the loop product, reminds one of Witten's paper [10]. There it was pointed out that closed string interactions looked at directly as in the Figure 1 are non-associative. To get around this, a marking point was introduced in [10] to facilitate the definition of an associative multiplication of (open) strings. Thus also our string bracket is non-associative but satisfies Jacobi and it arises from an associative product of loops (marked strings).

There is also a dictionary relating our constructions with those in algebra begun by Gerstenhaber [1].

In a sequel we will discuss a rich world of general string operations in the chains of the loop space. We find a structure like a big part of a two dimensional field theory associated to each manifold  $M^d$ . In particular we investigate the Lie<sub> $\infty$ </sub> structure described in Theorem 6.2 and Corollary 6.3 as well as co-versions. We also hope to follow Stasheff's specific suggestion to relate our structure to the work of the physicist Zwiebach [11] and [13].

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## 2 The loop product $\bullet$

We think of the circle  $\mathbb{S}^1$  as  $\mathbb{R}/\mathbb{Z}$ , so each point can specified by some  $x \in [0, 1)$ . Unless otherwise stated, M is an orientable manifold of dimension d. A loop in M is a continuous map from  $\mathbb{S}^1$  to M. Observe that a loop has a marked point: the image of 0. Map( $\mathbb{S}^1, M$ ), the space of all loops, will be denoted by  $\mathbb{L}(M)$ , or by  $\mathbb{L}$  since M is fixed throughout the discussion. By an *i*-chain we mean a linear combination of oriented *i*-dimensional families of loops in M. The parameter spaces of the families are taken from any standard list of cells closed under face operators.

In the algebraic topology of chains on a space there are two well known multiplications which we will combine to obtain a new structure. The first of these multiplications is the (transversal) intersection of chains in a *d*-manifold: an *i*-chain intersected with a *j*-chain gives an i + j - d-chain.

The second is the product of an *i*-chain of loops with a *j*-chain of loops, all of whose marked points are equal to some  $p \in M$ . Multiplying these chains yields an i + j cartesian product chain of composed loops with marked point at p in M.

Our new loop product  $\bullet$  is transversally defined at the chain level as follows (see Remark 2.2): given x, an *i*-chain of loops in M, and y, a *j*-chain

of loops in M, one first intersects the *i*-chain (of M) of marked points of x with the *j*-chain (of M) of marked points of y, to obtain an i + j - d chain c (of M) along which the marked point of x coincides with the marked point of y. Now we define the chain  $x \bullet y$ , by putting at each point of c the composed loop that first goes around the loop of x and then, around the loop of y (see Figure 2).



Figure 2: For  $M^3$ , the loop product of a 1-cell with a 2-cell is a 0-chain

In order to give the precise definition of the loop product we need the following remark.

**Remark 2.1** We will use the following **orientation convention** for our constructions. We will have a map of a domain cell, usually a product  $K_1 \times K_2 \times \ldots K_n$  into  $M \times M \times \ldots M$  (*l* factors) which is assumed to be transversal to a diagonal. We orient the product cell with the product of individual orientations. We orient the normal of the pull back by pulling back the orientation of the normal of the diagonal induced by some orientation of M. We then take the induced orientation on the pullback so that

(orientation on the pullback)(orientation on the normal) =

(product orientation of the cell domain)

Denote the (i - d) chains of  $\mathbb{L}$  by  $\mathbb{L}_i$  and the direct sum of all these by  $\mathbb{L}_*$ . Also, if  $x: K_x \longrightarrow \mathbb{L}$  is a cell, we denote by  $K_x$  its underlying set.

Now, for any pair of cells,  $x: K_x \longrightarrow \mathbb{L}, y: K_y \longrightarrow \mathbb{L}$  we define the set  $K_{x \bullet y}$ as the transversal preimage of the diagonal of  $M \times M$  under the map

$$K_x \times K_y \longrightarrow M \times M,$$

$$(k_x, k_y) \mapsto (x(k_x)(0), y(k_y)(0))$$

Now, define  $x \bullet y: K_{x \bullet y} \longrightarrow \mathbb{L}$  as

$$(x \bullet y)(k_x, k_y)(\gamma) = \begin{cases} x(k_x)(2\gamma) & \text{if } \gamma \in [0, \frac{1}{2}], \\ y(k_y)(2\gamma) & \text{if } \gamma \in [\frac{1}{2}, 1). \end{cases}$$

(we keep the notation  $K_{x \bullet y}$  because  $K_{x \bullet y}$  is a manifold which can be divided into cells). Orient  $K_{x \bullet y}$  by Remark 2.1. (Observe that with our new grading, if  $x \in \mathbb{L}_i$  and  $y \in \mathbb{L}_j$  then  $x \bullet y \in \mathbb{L}_{i+j}$ .

**Remark 2.2** We will adopt a point of view which is used in classical intersection theory of chains in a manifold. We say that a chain operation is *transversally defined* if it is defined for appropriately transversal cells.

We say that an identity between chain operations *holds transversally* if it holds on any finite subset of the chains where all constituents are appropriately transversal. Thus, the classical intersection is defined transversally at the chain level, is transversally associative and transversally graded commutative.  $\hfill \Box$ 

**Lemma 2.3** If  $x, y \in \mathbb{L}_*$  is a transversal pair then

$$\partial(x \bullet y) = \partial x \bullet y + (-1)^{|x|} x \bullet \partial y.$$

*Proof.* By definition, the underlying chain of  $x \bullet y$  is the oriented intersection chain in M of the marked points of x and of y respectively. Once the orientation of M is fixed, the orientations of intersections behave for calculations like orientations of normal directions.

Now, for each  $x \in \mathbb{L}_*$ , we denote by  $\widetilde{x}$  the chain of M of marked points of x. Thus, for transversal intersection one gets the familiar

$$\partial(\widetilde{x} \cap \widetilde{y}) = (\partial \widetilde{x} \cap y) + (-1)^{|x|} (\widetilde{x} \cap \partial \widetilde{y})$$

where  $|x| = \dim(x) - d$ .

Since  $K_{x \bullet y}$  is the underlying set of  $\widetilde{x} \cap \widetilde{y}$  we get

$$\partial K_{x \bullet y} = K_{\partial x \bullet y} + (-1)^{|x|} K_{x \bullet \partial y}.$$

The chain  $\partial(x \bullet y)$  is the restriction of  $x \bullet y$  to  $\partial K_{x \bullet y}$ . Therefore, the above formula yields the same formula for  $\bullet$ ,

$$\partial(x \bullet y) = \partial x \bullet y + (-1)^{|x|} x \bullet \partial y.$$

Let  $\mathbb{H}_i$  denote the *i*-th homology group of the loop space  $\mathbb{L}$  with the degree shifted down by *d* and set  $\mathbb{H}_{\star} = \mathbb{H}_{-d} \oplus \mathbb{H}_{-d+1} \oplus \ldots \mathbb{H}_0 \oplus \mathbb{H}_1 \ldots$  We refer to  $\mathbb{H}_{\star}$  as the *loop homology*.

**Corollary 2.4** The loop product • on chains passes to loop homology and defines a product

$$\mathbb{H}_i \otimes \mathbb{H}_j \xrightarrow{\bullet} \mathbb{H}_{i+j}$$

**Remark 2.5** The loop product is defined in all dimensions and may be nonzero way above the dimension of the manifold (e.g.,  $M = \mathbb{S}^2$ , Example 7.1, or  $\mathbb{S}^3$ , Corollary 3.6).

## 3 Associativity of ●, the \* operator, and homotopy commutativity of ●

Let us first discuss associativity. For the classical intersection product, we have that if three cycles are pairwise transversal, then the intersection product is literally associative at the chain level.

The classical based loop composition is associative up to homotopy (see Stasheff [7] and [8] for a complete discussion of this point).

Thus both of these classical chain products yield associative multiplications in homology, and thus we will have the same associativity for  $\bullet$ , the loop product, combining intersection and based loop composition.

**Proposition 3.1** The loop product in loop homology  $\mathbb{H}_i \otimes \mathbb{H}_j \xrightarrow{\bullet} \mathbb{H}_{i+j}$  is associative.

*Proof.* Assume that the three homology classes in question x, y, z are represented by cycles that are pairwise transversal. The intersection locus of  $(x \bullet y) \bullet z$  and  $x \bullet (y \bullet z)$  are literally equal with identical coorientations. The loop product is associative up to homotopy using the same considerations as in the based loop product, now parameterized by the points of the intersection  $x \cap y \cap z$ .

Let us turn to the question of commutativity. The intersection product of two transversal cycles is literally graded commutative. However, the based loop product (Pontryagin product) is often non-commutative even in homology. We will see that the • product, combining these two products is homotopy commutative at the chain level. One chain homotopy is given by a new binary operation x \* y defined for appropriately transversal pairs x, y in the following way. Consider the chain c (of M) where the marked point of x transversally intersects one of the images of the loops of y. Then at each point of c put the following loop: first go around the loop of y until the intersection point with the marked point of x. Then go around x and finally, go around the rest of y (see Figure 3).



Figure 3: The \* product

The precise definition is the following: let  $x: K_x \longrightarrow \mathbb{L}, y: K_y \longrightarrow M$  be two cells in  $\mathbb{L}_{\star}$ . Let  $K_{x*y}$  be the preimage of the diagonal of  $M \times M$  under the map

$$K_x \times [0,1] \times K_y \longrightarrow M \times M,$$
  
$$(k_x, s, k_y) \longmapsto (x(k_x)(0), y(k_y)(p(s))),$$

where  $p: [0, 1] \longrightarrow \mathbb{S}^1$  is the usual projection. Then  $x * y: K_{x*y} \longrightarrow \mathbb{L}$  is defined as

$$(x*y)(k_x, s, k_y)(\gamma) = \begin{cases} y(k_y)(2\gamma) & \text{if } \gamma \in [0, \frac{s}{2}).\\ x(k_x)(2\gamma - s) & \text{if } \gamma \in [\frac{s}{2}, \frac{s+1}{2}),\\ y(k_y)(2\gamma) & \text{if } \gamma \in [\frac{s+1}{2}, 1). \end{cases}$$

**Lemma 3.2** If  $x, y \in \mathbb{L}_{\star}$  are appropriately transversal, then

$$\partial(x*y) = \partial x*y + (-1)^{|x|+1}x*\partial y + (-1)^{|x|}(x \bullet y - (-1)^{|x||y|}y \bullet x).$$

*Proof.* Let  $\widetilde{x}$  denote the chain of M of the marked points of x and let  $\widetilde{y}$  denote the chain of M with parameter space  $[0,1] \times K_y$ , given by  $\widetilde{y}(s,k_y) = y(k_y)(p(s))$ , where  $p:[0,1] \longrightarrow \mathbb{S}^1$  is the usual projection.

Suppose that x and y are appropriately transversal. Since the underlying sets of x \* y and  $\tilde{x} \cap \tilde{y}$  coincide,  $\partial(x * y)$  is parametrized by the underlying set of  $\partial(\tilde{x} \cap \tilde{y})$ . Hence, for transversal intersection, one obtains

$$\partial(\widetilde{x} \cap \widetilde{y}) = (\partial \widetilde{x} \cap y) + (-1)^{|x|} (\widetilde{x} \cap \partial \widetilde{y}).$$

Since  $\partial([0,1] \times K_y) = \partial[0,1] \times K_y - [0,1] \times \partial K_y$ , the restriction of x \* y to the underlying set of  $(-1)^{|x|}(\widetilde{x} \cap \partial \widetilde{y})$  is  $(-1)^{|x|}(x \bullet y - (-1)^{|x||y|}y \bullet x) - (-1)^{|x|}x * \partial y$ . One the other hand, the restriction of x \* y to the underlying set of  $\partial \widetilde{x} \cap y$  is  $\partial x * y$ , which completes the proof.

In particular, if x and y are cycles, Lemma 3.2 implies that  $\partial(x * y) = \pm (x \bullet y - (-1)^{|x||y|} y \bullet x)$ . Therefore, we have

**Theorem 3.3**  $(\mathbb{H}_{\star}, \bullet)$  is an associative, (graded) commutative algebra.

Let us compare the loop product on loop homology  $(\mathbb{H}_{\star}, \bullet)$  with the usual homology of the manifold with intersection product  $(H_{\star}(M), \wedge)$  and the homology of the based loop space with the based loop product or Pontryagin product,  $(H_{\star}(\Omega), \times)$ . We have two maps,

$$H_{\star}(M) \xrightarrow{\varepsilon} \mathbb{H}_{\star} \xrightarrow{\cap} H_{\star}(\Omega)$$

where  $\varepsilon$  is the inclusion of constant (or even  $\varepsilon$  small) loops into all loops and  $\cap$  is the transversal intersection with one fiber of the projection loop space evaluation  $\xrightarrow{\text{evaluation}} M$ . If we use the usual grading on  $H(\Omega)$ , our shifted grading on  $\mathbb{H}_{\star}$ , the homology of the entire loop space, and the analogous shifted grading on  $H_{\star}(M)$ , then these products and the two maps have degree zero.

**Proposition 3.4**  $(H_{\star}(M), \wedge) \xrightarrow{\varepsilon} (\mathbb{H}_{\star}, \bullet) \xrightarrow{\cap} (H_{\star}(\Omega), \times)$  preserve products.

*Proof.* These follow directly from the definitions.

**Remark 3.5**  $\varepsilon$  is an injection onto a direct summand. For any Lie group manifold,  $\cap$  is a surjection.

**Corollary 3.6** For  $M = \mathbb{S}^3$ , the loop product is non-zero in infinitely many degrees.

*Proof.* By Remark 3.5,  $\mathbb{H}_{\star} \xrightarrow{\cap} H_{\star}(\Omega)$  is a surjection. With  $\mathbb{Q}$  coefficients, the homology of  $\Omega(\mathbb{S}^3)$  is the homology of  $\mathbb{C}P^{\infty}$  and the Pontryagin product gives a polynomial algebra on this generator in degree 2.

## 4 The loop bracket $\{,\}$ in loop homology

For easier thinking, let us now derive formulae by calculating in the chain complex of all homomorphisms  $\mathbb{L}_* \otimes \mathbb{L}_* \xrightarrow{\varphi} \mathbb{L}_*$  of degree  $-1, 0, 1, \ldots$ , with the usual  $\partial$  defined by the Leibniz rule with signs,

$$\partial(\varphi(a\otimes b)) = (\partial\varphi)(a\otimes b) + (-1)^{|\varphi|}\varphi(\partial(a\otimes b)),$$

 $\partial \varphi$ , which is defined by this relation, is also denoted  $[\partial, \varphi]$ , the graded commutator of  $\partial$  and  $\varphi$ .

If  $[\partial, \varphi] = 0$ ,  $\varphi$  is usually called a *chain map*. As an example, if  $\varphi(x \otimes y) = (-1)^{|x|} x * y$  of the previous section,  $[\partial, \varphi]$  evaluated on  $(x \otimes y)$  is

$$\begin{aligned} (-1)^{|x|}\partial(x*y) + ((-1)^{|x|+1}\partial x*y + x*\partial y) &= \\ (-1)^{|x|}(\partial(x*y) - \partial x*y - (-1)^{|x|+1}x*\partial y). \end{aligned}$$

By Lemma 3.2 this is equal to

$$x \bullet y - (-1)^{|x||y|} y \bullet x.$$

In other words, in the chain complex of homomorphisms  $\mathbb{L}_* \otimes \mathbb{L}_* \xrightarrow{\varphi} \mathbb{L}_*$ , the transversally defined 1-chain  $(-1)^{|x|}x * y$  has boundary the transversally defined zero chain  $(x \bullet y - (-1)^{|x||y|}y \bullet x)$ . Symbolically,

$$\partial *' = \bullet - \bullet \tau,$$

where  $x *' y = (-1)^{|x|} x * y$  and  $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$ .

Thus we consider  $*' + *'\tau$  and calculate it is a 1-cycle

$$[\partial, *' + *'\tau] = [\partial, *'] + [\partial, *'\tau] =$$
$$[\partial, *'] + [\partial, *']\tau = (\bullet - \bullet\tau) + (\bullet\tau - \bullet) = 0$$

using  $[\partial, \tau] = 0$  and  $\tau^2 = 0$ .

**Definition 4.1** The loop bracket  $\{x, y\}$  is defined transversally on  $\mathbb{L}_*$  by the formula

$$\{x, y\} = x * y - (-1)^{(|x|+1)(|y|+1)}y * x.$$

**Lemma 4.2** For  $x, y, z \in \mathbb{L}_*$  the associator of \* is symmetric in the first two variables,

$$x * (y * z) - (x * y) * z = (-1)^{(|x|+1)(|y|+1)} (y * (x * z) - (y * x) * z).$$

$$x^{*}(y^{*}z) = \underbrace{\begin{array}{c}y\\z\end{array}}_{z} + \underbrace{\begin{array}{c}y}_{z} + \underbrace{\end{array}{c}y}_{z} + \underbrace{\begin{array}{c}y}_{z} + \underbrace{\end{array}{c}y}_{z} +$$

Figure 4:  $K_{x*(y*z)}$  and  $K_{(x*y)*z}$ 

*Proof.* Observe that the parameter spaces of  $K_{x*(y*z)-(x*y)*z}$  and  $K_{y*(x*z)-(y*x)*z}$  consist in those values where the basepoint of the loops of x coincides with the one of the points of the loop of z and the basepoint of loops of y coincides with another. (see Figure 4)

Symbolically,

$$K_{x*(y*z)-(x*y)*z} = \{(k_x, s, k_y, t, k_z) \in K_x \times [0, 1] \times K_y \times [0, 1] \times K_z :$$
$$x(x_x)(0) = z(k_z)(s), y(k_y)(0) = z(k_z)(t)\}$$
$$K_{y*(x*z)-(y*x)*z} = \{(k_y, t, k_x, s, k_z) \in K_x \times [0, 1] \times K_y \times [0, 1] \times K_z :$$
$$x(x_x)(0) = z(k_z)(s), y(k_y)(0) = z(k_z)(t) = \}.$$

Clearly, there is a bijection between the underlying sets of the chains in question,  $K_{x*(y*z)-(x*y)*z}$  and  $K_{y*(x*z)-(y*x)*z}$ , which is orientation preserving if and only if  $(-1)^{(|x|+1)(|y|+1)} = 1$ .

**Proposition 4.3**  $(\mathbb{L}_{\star}, \{,\})$  is a graded Lie algebra (transversally) with all the degrees shifted by 1. In other words, for each  $x, y, z \in \mathbb{L}_{\star}$  mutually transversal,

$$\begin{array}{l} (1) \ \{x, \{y, z\}\} = \{x, \{y, z\}\} + (-1)^{(|x|+1)(|y|+1)} \{y, \{x, z\}\}. \\ (2) \ \{x, y\} = -(-1)^{(|x|+1)(|y|+1)} \{y, x\} \end{array}$$

*Proof.* Let us prove (1). By Lemma 4.2,

$$\begin{split} \{\{x,y\},z\}+(-1)^{(|x|+1)(|y|+1)}\{y,\{x,z\}\}-\{x,\{y,z\}\} = \\ (x*y)*z-(-1)^{(|x|+1)(|y|+1)}(y*x)*z-(-1)^{(|x|+|y|)(|z|+1)}z*(x*y)+(-1)^{|x||z|+|y||z|+|x||y|+1)}z*(y*x) \\ +(-1)^{(|x|+1)(|y|+1)}y*(x*z)-(-1)^{(|x|+1)(|y|+|z|)}y*(z*x) \end{split}$$

$$\begin{aligned} &-(-1)^{(|y|+1)(|z|+1)}(x*z)*y+(-1)^{(|x|+|y|)(|z|+1)}(z*x)*y\\ &-x*(y*z)+(-1)^{(|y|+1)(|z|+1)}x*(z*y)+(-1)^{(|x|+1)(|y|+|z|)}(y*z)*x\\ &-(-1)^{|x||y|+|y||z|+|x||z|+1}(z*y)*x=0 \end{aligned}$$

**Remark 4.4** An identical Proposition and proof can be found in [1] in a purely algebraic context.  $\Box$ 

**Corollary 4.5** The loop homology with the loop bracket,  $(\mathbb{H}, \{,\})$  is a graded Lie algebra of degree +1.

Now we discuss the compatibility of loop bracket and loop product. By Lemma 3.2, the map (transversally defined)

$$\mathbb{L}_{\star}\otimes\mathbb{L}_{\star}\otimes\mathbb{L}_{\star}\longrightarrow\mathbb{L}_{\star},$$

 $x_1 \otimes x_2 \otimes y \longrightarrow (x_1 \bullet x_2) * y - x_1 \bullet (x_2 * y) - (-1)^{|x_1|(|y|+1)} (x_1 * y) \bullet x_2$ 

is a chain map, i.e., the commutator with  $\partial$  is zero transversally.

Lemma 4.6 Let  $x, x_1, x_2, y, y_1, y_2 \in \mathbb{L}_*$  be appropriately transversal, then (1)  $x * (y_1 \bullet y_2) = (x * y_1) \bullet y_2 + (-1)^{|y_1|(|x|+1)}y_1 \bullet (x * y_2)$ (2)  $(x_1 \bullet x_2) * y - x_1 \bullet (x_2 * y) - (-1)^{|x_1|(|y|+1)}(x_1 * y) \bullet x_2 \simeq 0$ , where  $\simeq$ means chain homotopy.

*Proof.* The proof of (1) is easier because the equation holds transversally at the chain level: The set of parameters where the marked point of x coincides with one of the images of  $y_1 \bullet y_2$  is the union of the set of parameters where the marked point of x coincides with one of the image of  $y_1$  union the set of parameters where the marked point of x coincides with one of the image of  $y_1$  union the set of parameters where the marked point of x coincides with one of the image of  $y_2$ . (see Figure 5)

Let us prove (2). The idea (see Figure 6) is that the 2 dimensional chain

$$\varphi: \mathbb{L}_* \otimes \mathbb{L}_* \otimes \mathbb{L}_* \longrightarrow \mathbb{L}_*,$$
$$x_1 \otimes x_2 \otimes y \xrightarrow{\varphi} \varphi_{x_1, x_2, y}$$

where  $x_1$  and  $x_2$  attach to y at pairs of arbitrary points in such a way relative to the cyclic order that  $x_1$  is between the marked point and  $x_2$ , provides a chain homotopy between the two sides. More precisely, for each pair of points

$$(s,t) \in T = \{(s,t) \in [0,1] \times [0,1] : s+t \le 1\},\$$

$$\mathbf{x} \star (\mathbf{y}_1 \cdot \mathbf{y}_2) \equiv \underbrace{\mathbf{y}_1}_{\mathbf{y}_1} \underbrace{\mathbf{y}_2}_{\mathbf{y}_2} + \underbrace{\mathbf{y}_1}_{\mathbf{y}_2} \underbrace{\mathbf{y}_2}_{\mathbf{y}_2}$$

Figure 5: Proof of Lemma 4.6(1)

we will define a chain such that  $x_1$  is attached to y at t and  $x_2$  is attached to y at 1 - s.

Let

$$K = \{ (k_{x_1}, k_{x_2}, s, k_y, t) \in K_{x_1} \times K_{x_2} \times [0, 1] \times K_y \times [0, 1] :$$
$$x_1(k_{x_1})(0) = y(k_y)(t), x_2(k_{x_2})(0) = y(k_y)(s), s, t \in [0, 1]; s + t \le 1 \}$$
We define a map  $\varphi_{x_1, x_2, y} : K \longrightarrow \mathbb{L}$ .

$$\varphi_{x_1,x_2,y}(k_{x_1},k_{x_2},s,k_y,t)(\gamma) = \begin{cases} y(3\gamma) & \text{if } \gamma \in [0,\frac{t}{3}] \\ x_1(3\gamma-t) & \text{if } \gamma \in [\frac{t}{3},\frac{t+1}{3}]. \\ y(3\gamma) & \text{if } \gamma \in [\frac{t+1}{3},\frac{1}{3}] \\ x_2(3\gamma+s-1) & \text{if } \gamma \in [\frac{1-s}{3},\frac{2-s}{3}]. \\ y(3\gamma) & \text{if } \gamma \in [\frac{2-s}{3},1]. \end{cases}$$

for each  $(k_{x_1}, k_{x_2}, s, k_y, t) \in K, \gamma \in \mathbb{S}^1$ . Since the restriction of  $\varphi$  to

(i) 
$$K \cap \{(k_{x_1}, k_{x_2}, s, k_y, t) \in K_{x_1} \times K_{x_2} \times [0, 1] \times K_y \times [0, 1] : t = 0\}$$
 is  $x_1 \bullet (x_2 * y).$ 

(ii) 
$$K \cap \{(k_{x_1}, k_{x_2}, s, k_y, t) \in K_{x_1} \times K_{x_2} \times [0, 1] \times K_y \times [0, 1] : s = 1\}$$
 is  
 $(-1)^{x_2(y-1)}(x_1 * y) \bullet x_2$ 

(iii) 
$$K \cap \{(k_{x_1}, k_{x_2}, s, k_y, t) \in K_{x_1} \times K_{x_2} \times [0, 1] \times K_y \times [0, 1] : s + t = 1\}$$
 is  
 $(x_1 \bullet x_2) * y$ 

then

$$\partial K = K_{\partial y, x_1, x_2} + (-1)^y K_{y \partial x_1, x_2} + (-1)^{y + x_1} K_{y, x_1, \partial x_2} + K_{(x_1 \bullet x_2) * y} - K_{x_1 \bullet (x_2 * y)} - (-1)^{x_2} K_{(x_1 * y) \bullet x_2}.$$

 $\partial \varphi$  is the restriction of  $\varphi$  to  $\partial K$ . Thus

 $(\partial \varphi - \varphi \partial)(x_1 \otimes x_2 \otimes y) = (x_1 \bullet x_2) * y - x_1 \bullet (x_2 * y) - (-1)^{|x_1|(|y|+1)}(x_1 * y) \bullet x_2$ which completes the proof of (2). (See also [1]).



Figure 6: Proof of Lemma 4.6(2)

By Corollary 4.5 and Lemma 4.6 we have,

**Theorem 4.7** The loop product  $\bullet$  with the loop bracket  $\{,\}$  makes the loop homology into a Gerstenhaber algebra, namely:

- (1) The loop product  $\bullet$  defines a graded commutative, associative algebra.
- (2) {,} is a Lie bracket of degree 1, which means that for each  $a, b, c \in \mathbb{H}_*$ (i)  $\{a, b\} = -(-1)^{(|a|+1)(|b|+1)} \{b, a\}$ 
  - $(ii) \ \{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)} \{b, \{a, c\}\}\$
- (3)  $\{a, b \bullet c\} = \{a, b\} \bullet c + (-1)^{|b|(|a|-1)} b \bullet \{a, c\}.$

## 5 The $\triangle$ operator

Now we consider the degree +1 operation on the chains of the loop space,

$$\rightarrow \mathbb{L}_i \xrightarrow{\Delta} \mathbb{L}_{i+1} \rightarrow$$

given by the circle action on  $\operatorname{Map}(\mathbb{S}^1, M)$ . It can be defined in the following way: If  $x: K_x \longrightarrow \mathbb{L}$  is an *i*-chain then  $\Delta(x): \mathbb{S}^1 \times K_x \longrightarrow \mathbb{L}$  is the i+1 chain such that for each  $(s, k_x) \in \mathbb{S}^1 \times K_x$ ,  $\Delta(x)(s, k_x)(\gamma) = x(k_x)(\gamma + s)$ .

Since  $\Delta$  commutes with the  $\partial$  operator on chains, it passes to the loop homology, the homology of the free loop space, inducing a degree +1 operator

 $\Delta$ . Moreover, if x is an *i*-chain and  $k \geq 1$  then  $(\Delta)^k(x)$  has always geometric dimension i + 1. Therefore we obtain

**Proposition 5.1**  $\Delta: \mathbb{H}_{\star} \longrightarrow \mathbb{H}_{\star}$  is a degree +1 operator and  $\Delta \circ \Delta = 0$ .

We want to study how  $\Delta$  interacts with the above structure • and  $\{,\}$ . In order to do it, we need to define two auxiliary degree +1 operators on  $\mathbb{L}_*$ 

$$\Delta_1, \Delta_2: \mathbb{L}_* \longrightarrow \mathbb{L}_*.$$

Let  $x: K_x \longrightarrow \mathbb{L}$  be a k-chain. Then

$$\Delta_1(x): [0, \frac{1}{2}] \times K_x \longrightarrow \mathbb{L}, \qquad \Delta_2(x): [\frac{1}{2}, 1] \longrightarrow \mathbb{L},$$

are the k + 1-chains defined by

$$\Delta_1(x)(s,k_x)(\gamma) = x(k_x)(\gamma+s) \text{ and } \Delta_2(x)(s,k_x)(\gamma) = x(k_x)(\gamma+s).$$

Hence,  $\Delta = \Delta_1 + \Delta_2$ 

The transversally defined map

x

$$\mathbb{L}_{\star} \otimes \mathbb{L}_{\star} \longrightarrow \mathbb{L}_{\star},$$
$$x \otimes y \longrightarrow x \bullet \Delta y$$

is a chain map because it is composition of (transversally defined) chain maps. On the other hand, using Lemma 3.2 one can prove that the map

$$\mathbb{L}_{\star} \otimes \mathbb{L}_{\star} \longrightarrow \mathbb{L}_{\star},$$
$$\otimes y \longrightarrow (-1)^{|x|} \Delta_2(x \bullet y) - x * y$$

is also a chain map. Moreover, these two chain maps are chain homotopic, as is shown in the next lemma.

Lemma 5.2 For  $x, y \in \mathbb{L}_{\star}$ ,

$$(-1)^{|x|}\Delta_2(x \bullet y) - x * y \simeq x \bullet \Delta y$$

*Proof.* First, the idea of the proof: consider the chain operation

$$\varphi \colon \mathbb{L}_{\star} \otimes \mathbb{L}_{\star} \longrightarrow \mathbb{L}_{\star}$$

where the loop of x is attached to any point of the loop of y and one goes around a part of the loop of y, starting at any point between the marked



Figure 7: Proof of Lemma 5.2

points of y and x and ending where x is attached, then goes around the loop of x and finally, around the rest of y. (See Figure 7).

More precisely,

Let  $x: K_x \longrightarrow \mathbb{L}, y: K_y \longrightarrow \mathbb{L}$  be two cells of  $\mathbb{L}_*$ . Consider  $K_{x*y}$ , the parameter space of x \* y and set

$$K = \{ (t, k_x, s, k_y) \in [0, 1] \times K_{x*y} : 0 \le t \le s \le 1 \}$$

and define a chain  $\varphi: K \longrightarrow \mathbb{L}$  as follows

$$\varphi(t, k_x, s, k_y)(\gamma) = \begin{cases} y(k_y)(2\gamma + t) & \text{if } \gamma \in [0, \frac{s-t}{2}], \\ x(k_x)(2\gamma - s + t) & \text{if } \gamma \in [\frac{s-t}{2}, \frac{s-t+1}{2}], \\ y(k_y)(2\gamma + t) & \text{if } \gamma \in [\frac{s-t+1}{2}, 1]. \end{cases}$$

Observe that x is attached to y at the image of s by y and the image of t by y is the marked point of the resultant loop.

Proceeding in an analogous way as we did in the proof of Lemma 3.2, we obtain

$$(\partial \varphi - \varphi \partial)(x \otimes y) = (-1)^{|x|} \Delta_2(x \bullet y) - x * y - x \bullet \Delta y$$

as desired.

**Corollary 5.3** The loop bracket  $\{,\}$  on the loop homology is the deviation of  $\Delta$  from being a derivation of the loop product. In other words, for  $a, b \in \mathbb{H}$ 

$$\{a,b\} = (-1)^{|a|} \Delta(a \bullet b) - (-1)^{|a|} \Delta a \bullet b - a \bullet \Delta b$$

**Theorem 5.4** The loop product  $\bullet$  and the operator  $\Delta$  make the loop homoloqy into a Batalin Vilkovisky algebra, namely:

- (1) is a graded commutative associative algebra.
- (2)  $\Delta \circ \Delta = 0.$

(3) 
$$(-1)^{|a|}\Delta(a \bullet b) - (-1)^{|a|}\Delta a \bullet b - a \bullet \Delta b$$
 is a derivation of each variable.

*Proof.* By Theorem 4.7, Proposition 5.1 and Corollary 5.3.

**Remark 5.5** The alternative definition of a Batalin Vilkovisky algebra as a graded commutative algebra  $(A, \cdot)$  with a degree +1 operator  $\Delta: A \longrightarrow A$ such that  $\Delta \circ \Delta = 0$  and for each  $a, b, c \in A$ 

$$\Delta(a \cdot b \cdot c) = \Delta(a \cdot b) \cdot c + (-1)^{|a|} a \cdot \Delta(b \cdot c) + (-1)^{(|a|-1)b} b \cdot \Delta(a \cdot c)$$
$$-\Delta(a) \cdot b \cdot c - (-1)^{|a|} a \cdot \Delta(b) \cdot c - (-1)^{|a|+|b|} a \cdot b \cdot \Delta(c)$$
be found in Getzler [2].

can be found in Getzler |2|.

#### 6 The String Bracket

Using the circle action on the free loop space, we can define the equivariant homology  $\mathcal{H}_{\star}$  of the entire loop space. We could describe this as the ordinary homology of the quotient space  $\mathcal{S}$  of general smooth mappings  $(\mathbb{S}^1, M^d)$  by the circle action of rotation in the domain circle  $\mathbb{S}^1$ . The space  $\mathcal{S}$  can be viewed as the space of all general smooth closed curves in M. Thus we refer to the equivariant homology of the mapping space  $Map(\mathbb{S}^1, M)$  as string homology. The circle fibration

 $\mathbb{S}^1 \longrightarrow (\text{typical loops}) \longrightarrow \mathcal{S} = \text{string space}$ 

leads to an exact sequence (geometric grading)

$$\longrightarrow \mathbb{H}_i \xrightarrow{\mathrm{E}} \mathcal{H}_i \xrightarrow{c} \mathcal{H}_{i-2} \xrightarrow{\mathrm{M}} \mathbb{H}_{i-1} \longrightarrow \ldots$$

where E forgets the marked point of each member of a family of loops, M places a mark on each string in a family in all possible positions (and c is defined by cap product with the characteristic class of the circle bundle above).

The operator  $\Delta$  above is the composition  $M \circ E$ . The composition  $E \circ M$  on homology is zero, as part of the exactness above.

Any operation  $\mathbb{H}^{\otimes k}_{\star} \xrightarrow{\widetilde{\sigma}} \mathbb{H}_{\star}$  given by composition of  $\bullet$  and  $\Delta$  yields an operation  $\mathcal{H}^{\otimes k}_{\star} \xrightarrow{\sigma} \mathcal{H}_{\star}, \sigma = \mathbf{E} \circ \widetilde{\sigma} \circ \mathbf{M}^{\otimes k}$ .

In particular, taking  $\tilde{\sigma} = \bullet$  and adding a sign, gives the binary operation  $\mathcal{H}_{\star} \otimes \mathcal{H}_{\star} \xrightarrow{[.]} \mathcal{H}_{\star}$  called the *string bracket*,

$$[a,b] = (-1)^{|a|} \mathcal{E}(\mathcal{M}(a) \bullet \mathcal{M}(b)).$$

where |a| = dimension a - d.

**Theorem 6.1** String homology with the string bracket,  $(\mathcal{H}_{\star}, [,])$  is a graded Lie algebra of degree (2 - d) for the geometric grading.

*Proof.* By Theorem 3.3,  $\bullet$  is graded commutative. So, since M has degree +1,

$$[a,b] = (-1)^{|a|+(|a|+1)(|b|+1)} \mathcal{E}(\mathcal{M}(b) \bullet \mathcal{M}(a)) = -(-1)^{|a|\cdot|b|} [b,a].$$

To prove Jacobi, replace a (resp. b, c) by M(a) (resp. M(b), M(c)) in the Leibniz property (3) of Theorem 4.7 and apply E to both sides of the equation to obtain

$$E \left( \{ M(a), M(b) \bullet M(c) \} - \{ M(a), M(b) \} \bullet c - (-1)^{|a|(|b|+1)} M(b) \bullet \{ M(a), M(c) \} \right) = 0.$$

Now, use Theorem 5.4 and the fact that  $M \circ E = \Delta$  to replace in the above equation each of the brackets  $\{x, y\}$  by the formula  $(-1)^{|c|}(M \circ E)(c \bullet d) - (-1)^{|c|}(M \circ E)(c) \bullet d - c \bullet (M \circ E)(d)$ . Since  $E \circ M = 0$  we cancel the terms where  $E \circ M$  appear and so we obtain

$$\begin{split} & \mathbf{E} \left( -\mathbf{M}(a) \bullet \mathbf{M} \mathbf{E}(\mathbf{M}(b) \bullet \mathbf{M}(c)) - (-1)^{|a|+1} \mathbf{M} \mathbf{E} \left( \mathbf{M}(a) \bullet \mathbf{M}(b) \right) \bullet \mathbf{M} c \\ & - (-1)^{|a|(|b|+1)+|a|+1} \mathbf{M}(b) \bullet \mathbf{M} \mathbf{E} \left( \mathbf{M}(a) \bullet \mathbf{M}(c) \right) ) = 0 \end{split}$$

Now, replacing in the above formula each occurrence of E(M(d), M(e)) by  $(-1)^{|d|}[d, e]$  yields

$$-(-1)^{|a|+|b|}[a, [b, c]] + (-1)^{|a|+|b|}[[a, b], c] + (-1)^{|a||b|+|a|+|b|}[b, [a, c]] = 0.$$

Hence,

$$[a, [b, c]] = [[a, b], c] + (-1)^{(|a||b|)}[b, [a, c]].$$

By the above procedure, considering  $\mathbf{E} \circ \widetilde{\sigma} \circ \mathbf{M}^{\otimes k}$  we define operations

$$\bar{m}_k: \mathcal{H}^{\otimes k} \longrightarrow \mathcal{H}, \qquad k = 2, 3, 4, \dots$$

by the formula

$$\mathcal{H}^{\otimes k} \xrightarrow{M^{\otimes k}} \mathbb{H}^{\otimes k^{(\bullet)} \overset{\otimes (k-1)}{\longrightarrow}} \mathbb{H} \xrightarrow{\mathrm{E}} \mathcal{H}.$$

Note that if we shift the grading on string homology by (-d+1) from its geometric grading the degree of each  $\bar{m}_k$  becomes +1(=-(k-1)d+k-k(-d+1)+(-d+1)).

Now extend each  $\bar{m}_k$  to a coderivation  $m_k$  to  $\Lambda \mathcal{H}_{\star}$ . Here,  $\Lambda \mathcal{H}_{\star}$  is the free graded commutative coalgebra on  $\mathcal{H}_{\star}$  with the new *algebraic grading* and with the coalgebra structure which dualizes the usual  $\wedge$  algebra structure on  $\Lambda \mathcal{H}^{\star}$ . Thus  $m_k$  is the unique operation whose dual operation is the (unique) derivation on  $\Lambda \mathcal{H}^{\star}$  extending the operator dual to  $\bar{m}_k$ .

One knows the Jacobi identity for the string bracket  $\bar{m}_2$  is equivalent to the relation  $m_2 \circ m_2 = 0$  for the associated coderivation  $m_2$ .

The Jacobi relation for  $\bar{m}_2$  generalizes to the entire collection  $\{\bar{m}_2, \bar{m}_3, \ldots\}$  in the following way.

**Theorem 6.2** The associated coderivations  $\{m_2, m_3, \ldots\}$  of the free commutative coalgebra  $\Lambda \mathcal{H}_{\star}$  on the string homology  $\mathcal{H}_{\star}$  satisfy:

- (i)  $m_k \circ m_k = 0$ , for  $k = 2, 3, 4 \dots$
- (*ii*)  $m_k \circ m_r + m_r \circ m_k = 0$  for  $k, r = 2, 3, 4, \dots$

*Proof.* We only have to show the equations in the case when the range of the commutator being studied lies in monomial degree one. Since the commutator of coderivations is a coderivation, this is enough to show it is identically zero.

We illustrate the proof of (*ii*) for k = 3, r = 2. We have four families of closed curves  $A_1, A_2, A_3, A_4$ . Then  $(m_2 \circ m_3)(A_1 \wedge A_2 \wedge A_3 \wedge A_4)$  can be viewed a sum of twelve terms, each of them labeled with

$$(\{A_{i_1}, A_{i_2}, A_{i_3}\}, A_{i_1}),$$

where  $\{A_{i_1}, A_{i_2}, A_{i_3}\}$  runs over all possible choices of three families from  $\{A_1, A_2, A_3, A_4\}$ , with a preferred element.

Analogously,  $(m_2 \circ m_3)(A_1 \wedge A_2 \wedge A_3 \wedge A_4)$  can be viewed as a sum of twelve terms, each of them labeled with

$$(\{A_{l_1}, A_{l_2}\}, A_{l_1}),$$

where  $\{A_{l_1}, A_{l_2}\}$  runs over all possible choices of two families with a preferred element.

A correspondence of the two sets of labels is given by the map

$$(\{A_{i_1}, A_{i_2}, A_{i_3}\}, A_{i_1}) \longrightarrow (\{A_{i_1}, A_n\}, A_{i_1})$$

where  $A_n$  is the only family not in  $\{A_{i_1}, A_{i_2}, A_{i_3}\}$ .

Now, we will see that corresponding pair of terms appear in  $m_3 \circ m_2 + m_2 \circ m_3$  with different sign.

Consider, for instance, the corresponding pair of terms labeled by

$$(\{A_1, A_2, A_3\}, A_3)$$
 and  $(\{A_3, A_4\}, A_3)$ .

For simplicity, let us denote by  $A_i$  the parameter space of the family  $A_i$ . We can assume that each  $A_i$  is a cell, and that  $M(A_i)$  is a map

$$M(A_i): \mathbb{S}^1 \times A_i \longrightarrow \mathbb{L}$$

where

$$M(A_i)(s, a)(\gamma) = A_i(a)(s + \gamma).$$

The parameter space of the term labeled  $(\{A_1, A_2, A_3\}, A_3)$  is  $K_{\varphi}$ , the preimage of diag  $M^3 \times \text{diag} M^2$  under the map

$$\mathbb{S}^1 \times \mathbb{S}^1 \times A_1 \times \mathbb{S}^1 \times A_2 \times \mathbb{S}^1 \times A_3 \times \mathbb{S}^1 \times A_4 \stackrel{\varphi}{\longrightarrow} M^3 \times M^2,$$

given by  $\varphi(t, s_1, k_1, s_2, k_2, s_3, k_3, s_4, k_4) =$ 

$$((A_1(k_1)(s_1), A_2(k_2)(s_2), A_3(k_3)(s_3)), (A_3(k_3)(t), A_4(k_4)(s_4))))$$

The parameter space of the term labeled with  $(\{A_3, A_4\}, A_3), K_{\psi}$  is the preimage of diag $M^3 \times \text{diag}M^2$  under the map

$$\mathbb{S}^1 \times \mathbb{S}^1 \times A_1 \times \mathbb{S}^1 \times A_2 \times \mathbb{S}^1 \times A_3 \times \mathbb{S}^1 \times A_4 \stackrel{\psi}{\longrightarrow} M^3 \times M^2,$$

given by  $\psi(s_1, t, k_1, s_2, k_2, s_3, k_3, s_4, k_4) =$ 

$$((A_1(k_1)(s_1), A_2(k_2)(s_2), A_3(k_3)(s_3)), (A_3(k_3)(t), A_4(k_4)(s_4)))).$$

Over each point of these parameter spaces, the loops of the terms labeled with  $(\{A_1, A_2, A_3\}, A_3)$  and  $(\{A_3, A_4\}, A_3)$  are as in Figure 8.

Observe that the only difference between  $\varphi$  and  $\psi$  is that  $s_1$  and t are interchanged. This produces the difference of sign.



Figure 8: The loop of the terms labeled with  $(\{A_1, A_2, A_3\}, A_3)$  and  $(\{A_3, A_4\}, A_3)$ 

Now, we prove (i). We illustrate for k = 3. There will be five families,  $A_1, A_2, A_3, A_4, A_5$ .  $(m_3 \circ m_3)(A_1 \wedge A_2 \wedge A_3 \wedge A_4 \wedge A_5)$  is the sum of thirty terms, each of them labeled with

$$(\{A_{i_1}, A_{i_2}, A_{i_3}\}, A_{i_1})$$

where  $(\{A_{i_1}, A_{i_2}, A_{i_3}\}$  runs over all subsets of three elements of  $\{A_1, A_2, A_3, A_4, A_5\}$ . We group these terms in pairs with the following correspondence

$$(\{A_{i_1}, A_{i_2}, A_{i_3}\}, A_{i_1}) \leftrightarrow (\{A_{i_1}, A_{j_1}, A_{j_2}\}, A_{i_1})$$

where  $\{A_{j_1}, A_{j_2}\} = \{A_1, A_2, A_3, A_4, A_5\} \setminus \{A_{i_1}, A_{i_2}, A_{i_3}\}$ . As in the proof of case (ii), we can see that the parameter spaces corresponding to pairs cancel, so (i) holds.

**Corollary 6.3** There exists an uncountable family  $\{\delta_{\Lambda}\}$  of  $\text{Lie}_{\infty}$  structures on the string homology. Namely, for each  $\Lambda \subset \{2, 3, \ldots\}$ ,

$$\delta_{\Lambda} \colon \Lambda \mathcal{H}_{\star} \longrightarrow \Lambda \mathcal{H}_{\star} \text{ defined as } \delta_{\Lambda} = \sum_{\lambda \in \Lambda} m_{\lambda}$$

is a coderivation which satisfies  $\delta_{\Lambda} \circ \delta_{\Lambda} = 0$ .

*Proof.* By Theorem 6.2 each  $\delta_{\lambda}$  is a coderivation of  $\Lambda \mathcal{H}_{\star}$  of square zero and so determines (by definition) a Lie<sub> $\infty$ </sub> structure on  $\mathcal{H}_{\star}$ .

## 7 Examples

**Example 7.1** Let us unwrap the string bracket for  $M^d$  when d = 2. If the genus is greater than one, the equivariant homology of the loop space is concentrated in dimension zero (except for the component of the trivial loop whose higher homology we ignore at the moment).

The zeroth equivariant homology group,  $\mathcal{H}_0$ , is the vector space with basis  $\hat{\pi}$ , where  $\hat{\pi}$  denotes the set of free homotopy classes of loops in M.

Let us calculate the string bracket of two elements  $a, b \in \mathcal{H}_0$ . Let  $\rho, \sigma: \mathbb{S}^1 \longrightarrow M$  be two loops such that their free homotopy classes are a and b respectively.

We consider g two elements x, y of  $\mathbb{L}_{-1}$  determined by two maps

$$\mathbb{S}^1 \xrightarrow{x,y} \operatorname{Map}(\mathbb{S}^1, M^2)$$

given by

$$x(s)(\gamma) = \varrho(s+\gamma)$$
  $y(t)(\gamma) = \sigma(t+\gamma).$ 

Hence,  $Ma = \bar{x}$  and  $Mb = \bar{y}$ , where  $\bar{x}$  denotes the homology class of a cycle x.

Assume that x and y are transversal.

Thus,  $K_{x \bullet y} = \{(s, t) \in \mathbb{S}^1 \times \mathbb{S}^1 : \varrho(s) = \sigma(t)\}$ . By transversality, the set of intersection points of the curves  $\varrho$  and  $\sigma$ ,  $\varrho \cap \sigma$  is finite. Observe that it is precisely  $\{\varrho(s) : (s, t) \in K_{x \bullet b}\}$ .

The orientation of  $K_{x \bullet y}$  is given by endowing each of its points with a sign. For each  $(s,t) \in K_{x \bullet y}$ , this sign is precisely, the intersection index  $\epsilon(\varrho, \sigma, \varrho(s))$  of the loops  $\varrho$  and  $\sigma$  at  $\varrho(s)$ . Then

$$\mathbf{M}a \bullet \mathbf{M}b = \sum_{p \in \varrho \cap \sigma} \epsilon(\varrho, \sigma, p) (\varrho \sharp \sigma)_p,$$

where  $(\varrho \sharp \sigma)_p$  denotes the free homotopy class which contains the loop product of  $\varrho$  and  $\sigma$  at p. Hence,

$$[a,b] = \mathcal{E}(\mathcal{M}a \bullet \mathcal{M}b) = \sum_{p \in \varrho \cap \sigma} \epsilon(\varrho, \sigma, p) \mathcal{E}((\varrho \sharp \sigma)_p),$$

It seems remarkable that this formula is well defined in the vector space of components, it is skew symmetric, and satisfies Jacobi. Thus the string bracket becomes for d = 2 the formula discovered by Wolpert [12] and Goldman [3] which plays a role in the symplectic structure of the Techmuller space [12](because of Thurston's earthquakes) and the symplectic structure on the flat *G*-bundles for *G* a compact semisimple Lie group [3].

Indeed, trying to understand and generalize Goldman's work [3] lead us to the general theory above.  $\hfill \Box$ 

**Example 7.2** Now take d = 3 and consider a possibly twisted circle bundle over  $M^3$  with base a surface F of genus greater than one. It is interesting for the string bracket of  $M^3$  to consider equivariant homology (i.e., string homology) in dimension one. Cycles are generated by maps of torii into  $M^3$ . The projection of the torus to F is homotopic to a circle. So we only get interesting examples of torii in  $M^3$  by taking all the fibers in  $M^3$  over some circle in F. Two of these torii A and B can be put in transversal position by putting their projections a and b in F in transversal position. We see that the string bracket of A and B in  $M^3$  is the lift of the string bracket of a and b in F (as described in Example 7.1). So for these 3-manifolds the string bracket is just as non trivial as the string bracket on surfaces.

A similar discussion applies to Seifert fibrations over surfaces.

 $\Box$ 

## 8 Loop product and cap product

The cohomology algebra of a space (with cup product) acts on the homology of a space (called cap product) via the duality formula

$$\langle a \cap x, b \rangle = \langle a \cup b, x \rangle$$

where  $\langle , \rangle$  is the dual pairing between homology and cohomology,  $\cup$  is cup product, and  $\cap$  is cap product. To relate this structure to our loop product  $\bullet$ , consider the diagram

$$\operatorname{Map}(\mathbb{S}^1, M) \xleftarrow{\mathrm{C}} \operatorname{Map}(\operatorname{figure} \operatorname{eight}, M) \xrightarrow{1} \operatorname{Map}(\mathbb{S}^1, M) \times \operatorname{Map}(\mathbb{S}^1, M)$$

where c denotes the composition of loops and i is the natural inclusion.

**Definition 8.1** A pair of cohomology classes (a, A) (in the appropriate spaces) is called a *compatible pair* if  $i^*A = c^*a$ .

**Theorem 8.2** For each x, y homology classes and compatible pair of classes (A, a)

loop product  $(A \cap (x \otimes y)) = a \cap (x \bullet y)$ .

*Proof.* The loop product is the composition of intersection with image i (which is represented as a codimension d submanifold being the transverse image of the diagonal under the map

 $\operatorname{Map}(\mathbb{S}^1, M) \times \operatorname{Map}(\mathbb{S}^1, M) \xrightarrow{\operatorname{marked points}} M \times M)$ 

with the induced transformation of c in homology. The first process commutes with cap product with first A then with its restriction to Map(figure eight, M). The second process commutes with capping with a class in Map( $\mathbb{S}^1, M$ ) or with its pull back via c to Map(figure eight, M)

# 9 Appendix 1: $\mathbb{S}^2$ and other simply connected manifolds

Fibrations such as

based loop maps on  $\mathbb{S}^2 \longrightarrow \text{all loops in } \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ 

have algebraic models. For example,  $\mathbb{S}^2$  and based loops on  $\mathbb{S}^2$  are modeled by  $(0; x; y \dots) \xrightarrow{d} (0; 0; x^2)$  and  $(x; \bar{y}) \xrightarrow{d} (0; 0)$  respectively. The notation gives the generators in degrees  $1; 2; 3; \dots$  respectively and what the differentials are. The model is the free commutative algebra with those generators provided with a derivation d of square zero and degree 1 with the specified values.

The total space of the fibration has a model  $(\bar{x}; \bar{y}, x; y) \xrightarrow{d} (0; -2x\bar{x}, 0; x^2)$ .

There is also a derivation  $\Delta$  of degree -1 and square zero given by  $(\bar{x}; \bar{y}; x; y) \xrightarrow{\Delta} (0; 0; \bar{x}; \bar{y})$  and  $d\Delta + \Delta d = 0$  is true (and in fact determines d given  $\Delta$  and d on the base of the fibration.)

The models are cochain models. The obvious maps serve as cochain maps corresponding to the maps between space. The cohomology and induced transformations are derived accordingly [5].

The equivariant cohomology or string cohomology has model obtained by adding one closed generator u of degree 2

$$(\bar{x}; \bar{y}, x, u; y) \xrightarrow{d} (0; -2x\bar{x}, \bar{x}u, 0; x^2 + \bar{y}u)$$

where  $\bar{d}u = 0$  and  $\bar{d}z = dz + (\Delta z)u$  determines the rule for the other generators z.

Calculating with these models we find the Betti numbers of the loop homology are all 1 in each dimension  $1, 2, 3, \ldots$  and the image of based loop homology in free homology is in degrees  $1, 3, 5, \ldots$ . Thus exactness for  $\mathbb{S}^2$  of

based loop homology  $\xrightarrow{\text{inclusion}}$  loop homology  $\xrightarrow{\cap}$  based loop homology

implies  $\cap$  has non trivial image in the even degrees  $0, 2, 4, \ldots$ . This shows the loop product is non-trivial for  $\mathbb{S}^2$ .

Calculating further, one finds  $\Delta$  is an isomorphism on loop homology from odd to even dimensions and is zero in even dimensions. Since  $\Delta = M \circ E$ , one finds that E is non zero in odd dimensions and M is non zero starting in odd dimensions. Since  $E \circ M = 0$ , one finds E is zero in even dimensions and M is zero starting in even dimensions. Thus the string bracket  $[x, y] = \pm E(M(x) \bullet M(y))$  is zero in all dimensions for  $\mathbb{S}^2$ .

## 9.1 General simply connected manifolds

This pattern works in general for simply connected manifold to describe the algorithm for calculating the rational homology of these spaces:

If M has minimal model (where each differential is quadratic + higher order terms)

$$(0; x_1, x_2, \ldots; y_1, y_2, \ldots) \xrightarrow{d} (dx_1, dx_2, \ldots; dy_1, dy_2, \ldots)$$

then the based loops on M has a model

$$(\bar{x_1}, \bar{x_2}, \ldots; \bar{y_1}, \bar{y_2}, \ldots) \xrightarrow{d} (0, 0, \ldots, 0; 0, \ldots)$$

and the free loop space with operations  $\Delta$  and d of degree -1 and 1 satisfying  $d\Delta + \Delta d = 0$  has model with generators

$$(\bar{x_1}, \bar{x_2}, \ldots; \bar{y_1}, \bar{y_2}, \ldots, x_1, x_2, \ldots; y_1, y_2, \ldots)$$

 $\Delta$  is defined by  $x_i \mapsto \bar{x}_i, y_i \mapsto \bar{y}_i$  and  $\bar{x}_i, \bar{y}_i \mapsto 0$ . Then d is defined so that  $d\Delta + \Delta d = 0$  and d is given as before on the  $x_i, y_i \dots$  coming from  $M^d$ . Thus  $(d\Delta + \Delta d)x_i = 0$  implies  $d\bar{x}_i = -\Delta(dx_i)$  can be calculated since  $dx_i$  and  $\Delta$  are known.

A equivariant model is obtained by adding to a free loop space model one more variable u in degree two with du = 0 and  $\bar{d}z = dz + \Delta(z) \cdot u$  for the other generators z [5].

**Remark 9.1** In [6] it was shown the ranks of the loop homology are unbounded for simply connected manifolds unless the minimal model has only one or two generators (like  $\mathbb{S}^2$  or  $\mathbb{S}^3$ ).

# 10 Appendix 2: $M^3$ and $K(\pi, 1)$ manifolds

It is known that any closed 3-manifold is a connected sum  $M_1 # M_2 # M_3 # \dots # M_n$ along  $\mathbb{S}^2$  where each of the  $M_i$ 's is of one of the following types

- (i)  $\pi_1$  is finite so the universal cover is homotopy equivalent to  $\mathbb{S}^3$ .
- (ii)  $M_i$  is  $\mathbb{S}^1 \times \mathbb{S}^2$ .
- (iii)  $\pi_1$  is infinite and the universal cover is contractible [4]

The technique of models plus finite group invariance can be used to treat the examples of type (i) and (ii).

If we treat examples of type (iii) it seems plausible one could develop an algorithm for the connected sum using our knowledge of  $\mathbb{S}^2$  and free product ideas.

We discuss type (iii) under the hypothesis of Thurston's geometrization picture.

- (1) If  $M_i$  is closed hyperbolic, each centralizer of a non-zero conjugacy class is infinite cyclic. Thus that component of the free loop space is a homotopy circle. By dimension reasons all loop products between these components are zero. The loop product reduces to the classical intersection product.
- (2) Otherwise M would be a union along torii of Seifert fibrations over surfaces with boundary and finite volume hyperbolic manifolds with neighborhoods of the cusps deleted.
- (3) If any non trivial Seifert fibrations are present, we have a rich structure of loop product as described in Example 7.2.
- (4) Finally, M could be a union along torii of hyperbolic pieces and we haven't analyzed these cases.

## **10.1** General $K(\pi, 1)$ manifolds, $d \ge 3$

The loop space of M is homotopy equivalent to a union over conjugacy classes  $\alpha$  in  $\pi$  of  $K(\pi_{\alpha}, 1)$  homotopy types, where  $\pi_{\alpha}$  is the centralizer of a representative of  $\alpha$ . In particular a component of  $\mathbb{L}(M)$  is homotopy equivalent to a covering space of M. Thus its homological dimension is at most d. Again for hyperbolic manifolds, the string bracket is zero for dimension reasons.

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