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LACUNARY SERIES AS QUADRATIC DIFFERENTIALS IN CONFORMAL DYNAMICS

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Introduction

Given a polynomial-like mapping P acting on a plane domain, we denote the forward iterates of P under composition by P^n. These iterates determine a dynamical system acting on a neighborhood of the Julia set. If we assume the Julia set is connected, we may uniformize the complement of the Julia set by a Riemann mapping r, mapping the complement to the exterior of the unit disk. The forward iterates P^n are conjugated to iterates $r \circ P^n \circ r^{-1}$, which forms a dynamical system acting on the exterior of the unit disk and which extends to an action on the unit circle. This conjugated dynamical system is expanding along the unit circle, which is a repeller.

The chief result of this paper is that such dynamical systems, for a given degree, form part of a Teichmüller space. It is an infinite dimensional complex manifold modeled on a Banach space and it has a Teichmüller's metric which is complete. The Banach space is a space of quadratic differentials and, for a special case, this space is identifiable with certain lacunary series. There is an associated infinite dimensional Banach space for every polynomial-like

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mapping and these Banach spaces generalize lacunary series. The homogeneity of smoothness of lacunary series and of more general vector fields for the dynamical system can be seen as a form of Mostow rigidity.

In order to create this complete Teichmüller space, we pick out a natural class which is larger than the class of polynomial-like mappings. It is the class *UAC of uniformly asymptotically conformal dynamical systems* with a fixed topological type, acting on an open neighborhood of the repeller and factored by asymptotically conformal equivalence.

These concepts are defined in section 1. For now we say only that there is a type of uniform almost isometric behaviour, with respect to the Poincaré metric, satisfied by the forward powers of a polynomial-like mapping P. This almost isometric behaviour is shared by elements of UAC. That polynomial-like mappings have almost isometric behaviour is the content of the *hyperbolic distortion lemma*, (Lemma 3). The lemma shows that single-valued branches of an P^n are nearly isometric and, in this sense, the dynamical system for P^n resembles a Fuchsian group. Of course, the transformations of a Fuchsian group are exact isometries.

Teichmüller's metric on the factor space T of UAC becomes the dynamical boundary dilatation metric, a metric which is defined in [4]. This metric identifies two elements of UAC if they have the same behaviour at fine scales; in particular, if two elements are identified, the limits of their scaling ratios will be identical. Elements of T have well-defined eigenvalues at repelling periodic fixed points. The fibers of the tangent space to T are spaces of holomorphic quadratic differentials for analytic realizations of the quasiconformal dynamical system. The fibers over the special polynomial-like maps of the form $P(z) = z^d$ are given by certain lacunary series. Our proof of the existence of fibers over the other polynomial-like mappings depends on a discussion of Bers' $\mathcal{L}$-operator [1] and another distortion lemma, called the *Schwarzian distortion lemma*, (Lemma 7). Its proof is a familiar chain rule and geometric series argument.

There is a Riemann surface lamination associated to the expanding mapping P. It is obtained from the germ of the action of the iterates of P on small open neighborhoods of the repeller, which is the intersection of the descending sequence of sets, $P^{-n}(V)$. Elements of the lamination are tails of strings of points $(..., z_3, z_2, z_1, z_0)$ which have the property that z_0 is in V-{the repeller} and $P(z_{n+1}) = z_n$ for each n. This space can be viewed locally as the direct product of a Cantor set with complex disks. The global form of the space is determined by the dynamics of P and we call it the Riemann surface lamination L for the mapping P. The theory of the Teichmüller space for a general Riemann surface lamination all of whose leaves are hyperbolic is developed in [6]. That theory encompasses the Teichmüller theory we consider here for UAC systems and some of the lemmas used in this article could be deduced from the more general theory.

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Section 1. Uniformly asymptotically conformal dynamical systems

Let Δ^* be the set of points z in complex plane for which $|z| > 1$ and let U be an arbitrary annular neighborhood in Δ^* of the unit circle. By the annular neighborhood U we mean a doubly connected open subset of Δ^* for which there are numbers r_1 and r_2 greater than 1 such that $\{z : 1 < |z| < r_1\} \subseteq U \subseteq \{z : 1 < |z| < r_2\}$. As a matter of technical convenience, we assume that P is defined on all of Δ^*, but the ingredients of P important to us depend only on its values on small annular neighborhoods of the unit circle.

We make the following assumptions on P. It is a quasiregular mapping defined on Δ^*, it is expanding of degree d in an annular neighborhood of the unit circle and it is *uniformly asymptotically conformal.*

By quasiregular we mean that P factors into a quasiconformal homeomorphism from Δ^* onto Δ^* followed by a holomorphic mapping.

By expanding of degree d in an annular neighborhood of the unit circle, we mean that there are annular neighborhoods U_1 and U_0 with $U_1 \subset U_0$ such that $U_0 - U_1$ is a topological annulus with positive modulus and such that P restricted to U_1 is an unbranched degree d covering of U_0. We also assume that the sets $U_n = P^{-n}(U_0)$ form a descending chain of annuli $U_0 \supset U_1 \cdots \supset U_n \cdots$ whose intersection is empty and that each ring $w_n = U_n - U_{n+1}$ is an annulus of positive modulus surrounding the unit circle. From the geometric neighborhood lemma (Lemma 5), the rings w_n are contained in $\{z : 1 < |z| < 1 + \varepsilon\}$ where ε converges to zero exponentially as n converges to ∞. The family of rings w_n related to P in this way is called a sequence of ring domains for P.

By *uniformly asymptotically conformal*, we mean that the branches of P^{-n} are nearly conformal on sufficiently small annular neighborhoods for all nonnegative integers n. More precisely, for every $\varepsilon > 0$, there is an annular neighborhood U, such that for every integer $n \geq 0$ and every z in $P^{-n}(U)$, the dilatation of P^n at z is less than $1 + \varepsilon$.

Definition. *Let UAC be the set of all mappings P which are quasiregular and defined on Δ^*, which are expanding and of degree d in an annular neighborhood of the unit circle and which are uniformly asymptotically conformal.*

The most obvious examples of degree 2 mappings P in UAC are $P(z) = z^2$ or $P(z) = z\left(\frac{z-a}{1-\bar{a}z}\right)$ with $|a| < 1$. Other examples can be obtained by conjugating the action of a polynomial-like mapping on the exterior of the filled-in Julia set to an action on the exterior of the unit circle. We will eventually see that there are elements of UAC which are not in the same conjugacy class as a polynomial-like map.

Let P and $\hat{P}$ be elements of UAC. Then there are annular neighborhoods U_1 and U_0 for P and $\hat{U}_1$ and $\hat{U}_0$ for $\hat{P}$ on which P and $\hat{P}$ are unbranched degree d coverings. By trimming the boundaries of $U_1, U_0, \hat{U}_1$ and $\hat{U}_0$, we can assume that the outer boundaries of these annular neighborhoods are quasicircles. Then we can construct a quasiconformal homeomorphism f from ω_0 to $\hat{\omega_0}$ such that $f \circ P(z) = \hat{P} \circ f(z)$ for values of z on the inner boundary of ω_0. By pulling back the quasiconformal mapping f defined in ω_0, the conjugacy f extends to a quasiconformal homeomorphism f from $\omega_1 \cup \omega_0$ onto $\hat{\omega_1} \cup \hat{\omega_o}$ satisfying $f \circ P(z) = \hat{P} \circ f(z)$ and which has maximal dilatation bounded by the product of the dilatation of f restricted to ω_0 and of the dilatations of P and $\hat{P}$. In a similar manner, f can be extended to the union of all the ω_k's satisfying the relation $f \circ P(z) = \hat{P} \circ f(z)$. We call a homeomorphism f constructed in this manner a pullback homeomorphism. The hypotheses that the backwards branches of P^n and $\hat{P}^n$ uniformly asymptotically conformal implies, in particular, that they are uniformly quasiconformal and, thus, the pullback homeomorphism f is quasiconformal on U_0.

We see that any two elements in UAC are conjugate in some annular neighborhood by a quasiconformal homeomorphism f. The conjugating mapping f is called asymptotically conformal if, for every $\varepsilon > 0$, there exists a number $r > 1$, such that the dilatation of f in the annulus $1 < |z| < r$ is less that $1+\varepsilon$. By demanding that f be asymptotically conformal, we obtain an equivalence relation on UAC.

Definition. *Two elements P and $\hat{P}$ in UAC are equivalent if there is a quasiconformal homeomorphism f asymptotically conformal at the boundary of Δ^* and defined on some annular neighborhood U such that $f \circ P(z) = \hat{P} \circ f(z)$ for all z with $|z| = 1$. The Teichmüller space T of uniformly asymptotically conformal expanding degree d mappings is the space UAC factored by this equivalence relation.*

Remarks. 1. Every element P of UAC can be symmetrized by Schwarz reflection. It becomes a quasiregular mapping P of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ which fixes the unit circle and is invariant under conjugation by the reflection $j(z) = 1/\overline{z}$. Therefore, it makes sense to speak of the values of $P(z)$ and $f(P(z))$ for values of z on the unit circle.

2. Even though mappings P in UAC are not differentiable, if q is a periodic point of P of order m on the unit circle, then the notion of the eigenvalue of P at q is well-defined on the equivalence class of P in T. Since P^m fixes q, it must map a sufficiently small disk neighborhood of q over itself leaving an intervening annulus. The hypothesis that P^m is asymptotically conformal implies that the modulus of this annulus, viewed as a function of the selected disk neighborhood, has a limit as the disk neighborhood shrinks towards the point q. If we define the eigenvalue of P at the periodic point q to be the exponential function applied to 2π times this limiting modulus, then it coincides with the usual notion of eigenvalue, namely, $\frac{dP^m}{dx}(q)$, when P is holomorphic at q.

Choose an element P in UAC and annular neighborhoods U_1 and U_0 such that P is a degree d, unbranched cover of U_1 over U_0. Define $QS(P)$ to be the set of quasiconformal homeomorphisms h mapping U_0 onto annular neighborhoods of the boundary of the unit disk such that the dynamical system $h \circ P^n \circ h^{-1}$ is uniformly asymptotically conformal. The Beltrami coefficient of a mapping h, which we denote by the symbol Beltr(h), is defined to be $\frac{\partial h}{\partial \overline{z}} / \frac{\partial h}{\partial z}$.

For h in $QS(P)$, we consider three different Beltrami coefficients:

i) the Beltrami coefficient μ of h,

ii) the Beltrami coefficient σ_n of $h \circ P^n$,

iii) the Beltrami coefficient $\mu_n = P^{n^*}\mu(z)$ defined by

$$\mu_n(z) = \mu(P^{n^*}(z))\overline{q}/q \quad \textit{where} \quad q = \frac{\partial}{\partial z}P^n(z).$$

Lemma 1. *(P-equivariant deformations) Assume h is a quasiconformal homeomorphism of* Δ^*. *The following conditions on h are equivalent:*

i) *h is an element of* $QS(P)$,

ii) *for every* $\varepsilon > 0$, *there exists an annular neighborhood U such that for all positive integers n and for all z in* $P^{-n}(U)$,

$$|\sigma_n(z) - \mu(z)| < \varepsilon.$$

iii) *for every* $\varepsilon > 0$, *there exists an annular neighborhood U such that for all positive integers n and for all z in* $P^{-n}(U)$,

$$|\mu_n(z) - \mu(z)| < \varepsilon.$$

Proof. The Beltrami coefficient of $(h \circ P^n) \circ h^{-1}$ is $\sigma_n - \mu$ divided by a term which is bounded away from zero. Therefore, the assertion that the dilatation of $h \circ P^n \circ h^{-1}$ is uniformly near to 1 for z in $P^{-n}(U)$ is equivalent to ii). Let ν_n be the Beltrami coefficient of P^n. Observe that μ_n, ν_n and σ_n are related by the equation

$$\sigma_n = \frac{\nu_n + \mu_n}{1 + \overline{\nu_n}\mu_n}$$

and, therefore,

$$\sigma_n - \mu_n = \frac{\nu_n - \mu_n^2\overline{\nu_n}}{1 + \overline{\nu_n}\mu_n}.$$

Since we can make $|\nu_n(z)|$ as small as we like by suitably choosing U and letting z be in $P^{-n}(U)$ and since $|\mu_n(z)|$ is uniformly bounded for these z, the equivalence of i) and iii) is a consequence of this last formula. □

We denote by $M(P)$ the space of Beltrami coefficients of mappings h in $QS(P)$. By the preceding lemma any element of $M(P)$ is a measurable complex valued function defined on U_0 satisfying the following properties:

a) $\text{esssup}_{z \text{ in } U_0} |\mu(z)| < 1$

b) for every $\varepsilon > 0$, there exists an annular neighborhood U of the boundary of the unit disk, such that for all positive integers n and all z in $P^{-n}(U)$, $|P^{n^*}\mu(z) - \mu(z)| < \varepsilon$.

Conversely, if μ satisfies these two properties, then by solving the Beltrami equation for a mapping with Beltrami coefficient μ, we obtain a mapping h such that $h \circ P \circ h^{-1}$ is uniformly asymptotically conformal.

Let S be the group of quasiconformal homeomorphisms of Δ^* onto Δ^* which are asymptotically conformal. This group determines an equivalence relation on $QS(P)$ by declaring two elements h_1 and h_2 of $QS(P)$ to be equivalent if there is an element s of S such that $s \circ h_1(z) = h_2(z)$ for $|z| = 1$. We let $QS(P)/S$ be the space of equivalence classes for this equivalence relation.

There is a mapping from $QS(P)$ to UAC; for any h in $QS(P)$, the mapping is $h \mapsto \hat{P}$ where $\hat{P} = h \circ P \circ h^{-1}$. By the pullback construction, we see that this mapping is surjective and, since T is a quotient space of UAC, we obtain a mapping from $QS(P)$ onto T. Take two elements of h_0 and h_1 of $QS(P)$ which conjugate P into P_0 and P_1, respectively. Assume that h_0 and h_1 are equivalent modulo S. Then there is an asymptotically conformal mapping s in S such that $s \circ h_0(z) = h_1(z)$ for $|z| = 1$. Then $s \circ P_0 \circ s^{-1}(z) = P_1(z)$ for $|z| = 1$ and, consequently, P_0 is equivalent to P_1 in UAC.

Conversely, if P_0 and P_1 are equivalent in UAC, there is an asymptotically conformal mapping s such that $s \circ P_0 \circ s^{-1}(z) = P_1(z)$ for all z on the boundary of the unit circle. If h_0 and h_1 conjugate P to P_0 and P_1, respectively, then we find that $s \circ h_0 \circ P^n \circ (s \circ h_0)^{-1} = h_{-1} \circ P^n \circ h_1^{-1}$ for all positive integers n. Since repelling periodic points of P are dense in the unit circle, we conclude that $s \circ h_0(z) = h_1(z)$ for z on the boundary of the unit circle and, hence, h_0 and h_1 are in the same equivalence class for $QS(P)/S$. We have proved the following theorem.

Theorem 1. *The natural mapping from $QS(P)$ onto T factors to an isomorphism from $QS(P)/S$ to T.*

Theorem 2. *The boundary dilatation metric on $QS(P)/S$ makes T into a complete metric space and the metric is independent of the base point P.*

Proof. In [5] it is shown that the boundary dilatation metric on QS $modS$

is complete. The issue here is to show that $QS(P)$ is a closed subset of QS. Assume that h is a quasiconformal mapping and that h_j is a sequence of quasiconformal mappings such that $||Beltr(h_j) - Beltr(h)||_\infty < \varepsilon$ for sufficiently large j. It follows that for z in $P^{-n}(U_0)$,

$$|Beltr(h_j \circ P^n)(z) - Beltr(h \circ P^n)(z)| < \varepsilon',$$

where $\varepsilon' = \varepsilon/(1-k^2)$ and k is a uniform bound on the absolute values of the Beltrami coefficients of P^n. On the other hand, since each h_j is in $QS(P)$, we know that for every $\varepsilon > 0$, there exists j_0 , such that for all $j \geq j_0$ and for all z in $P^{-n}(U_j)$,

$$|Beltr(h_j \circ P^n)(z) - Beltr(h_j)(z)| < \varepsilon.$$

Therefore, for all n and z in $P^{-n}(U_{j_0})$,

$$|Beltr(h \circ P^n)(z) - Beltr(h)| < 2\varepsilon + \varepsilon'.$$

The fact that the metric on $QS(P)/S$ is independent of base point follows from the fact that composition on the right induces an isometry for the respective Teichmüller metrics. □

Section 2. Polynomial-like mappings

An element of UAC which is conformal in some annular neighborhood U of the boundary of Δ^* is called polynomial-like. First we prove that polynomial-like mappings are dense in T.

Given an element P in UAC and a positive number ε, select annular neighborhoods U_1 and U_0 such that P restricted to U_1 is a degree d cover over U_0 and the dilatation of P on U_1 is less than $1+\varepsilon$. As before, we let $U_n = P^{-n}(U_0)$. Let μ be the Beltrami coefficient of P in U_n and let μ be identically zero outside of U_n. Form the quasiconformal homeomorphism g of Δ^* which fixes ∞ and which has Beltrami coefficient μ. The mapping g is conformal in $|z| > 1+\delta$ provided that the circle of radius $1+\delta$ contains U_n. Moreover, g is $1+\varepsilon$-quasiconformal in $|z|>1$. Since δ approaches zero as n approaches ∞, by the Hölder continuity of g, we may choose n large enough so that $g(U_1)$ is contained in U_0 and so that the complement of $g(U_1)$ in U_0 is an annulus of positive modulus.

Let $\hat{P} = P \circ g^{-1}$ and $\hat{U}_1 = g(U_1)$. Then $\hat{P}(U_1) = P(U_1) = U_0$ and P is a degree d unbranched covering of U_1 over U_0 and the closure of $\hat{U}_1$ is compact in U_0. Moreover, $\hat{P}$ is conformal in $\hat{U}$. Therefore, $\hat{P}$ is a polynomial-like element of UAC whose distance in the Teichmüller metric on $QS(P)/S$ is less than $log(1+\varepsilon)$ from P. We obtain the following result.

Lemma 2. (*Density of polynomial-like mappings*) *The set of polynomial-like mappings is dense in the Teichmüller space of uniformly asymptotically expanding degree d mappings with Teichmüller's boundary dilatation metric.*

The Poincaré metric for Δ^* is $\lambda(z)|dz|1 = \Big(|dz|/|z|log|z|\Big)$. We are concerned only with how this metric measures the sizes of objects which are very near to the boundary of the unit circle. If $\delta(z)$ is the distance from z to the boundary, the formula for $\lambda(z)$ is asymptotically equal to $(1/\delta(z))$ for values of z with $|z|$ near to 1. The Poincaré metric for $\Delta^* \cup \{\infty\}$ is $\Big(2|dz|/(|z|^2-1)\Big)$ and this has the same asymptotic values for $|z|$ near to 1. For the purposes of the next lemma, we could use either one of these two metrics. The lemma says that if P is polynomial-like, the branches of P^n, restricted to neighborhoods sufficiently near to the boundary, are approximate isometries in the Poincaré metric. It is a kind of distortion lemma, because it gives a bound on distortion for branches of P^n, which is independent of n.

Lemma 3. (*Hyperbolic Distortion*) *Assume P is a polynomial-like mapping acting on Δ^*. Then for every ε, there exists an annular neighborhood U, such that for every positive integer n and every z in $P^{-n}(U)$,*

$$\frac{1}{1+\varepsilon} \leq \frac{\lambda(P^n(z))|P^{n'}(z)|}{\lambda(z)} \leq 1+\varepsilon.$$

Proof. (This proof was explained to me by Mitsu Shishikura.) One views $\Delta^* - \{\infty\}$ as a half-cylinder, the upper half plane factored by the cyclic group generated by $z \mapsto z+1$. Let $\lambda(z)$ by the Poincaré metric for this half plane. One lifts the mapping P restricted to U_1 to a mapping $\hat{P}$ defined on a periodic strip S_1 in the upper half plane bounded by the real axis and a periodic curve which is the lift of the outer boundary of U_1. This lifted mapping maps S_1 over itself and onto a domain S_0 which is the lift of the domain U_0. $\hat{P}$ restricted to S_1 is one-to-one and $\hat{P}^{-1}$ is a mapping from the strip S_0 into itself. The natural extension of $\hat{P}^{-1}$ acts as an isometry of a hyperbolic plane. Let ρ be the hyperbolic metric for this plane. This hyperbolic plane contains the strip S_0. Let λ_0 be the hyperbolic metric for S_0. The inequalities $\lambda < \lambda_0$ and $\rho < \lambda_0$ follow from Schwarz's inequality. On the other hand by shrinking to strips smaller than S_0 whose boundaries are straight lines, we can use the exact formulas for the Poincaré metrics to deduce the following fact; given any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $z = x + iy$ with $0 < y < \delta$, $(1+\varepsilon)^{-1} \leq \rho(z)|dz|/\lambda(z)|dz| \leq 1+\varepsilon$. Because the natural extension of $\hat{P}^{-1}$ is a non-Euclidean isometry for the ρ-metric, we obtain the uniformity condition of the lemma. □

Section 3. Quadratic differentials for polynomial-like mappings

For the purposes of clarity of exposition, in this section we assume the degree d of our mappings is two; the case for general degree follows by trivial modifications, usually no more complicated than the replacement of the symbol 2 by d.

The first requirement is to construct a set of domains for the dynamical system generated by the branches of the mappings P^n which are analogous to a fundamental domain for a Fuchsian group. By assumption, P is a cover of a domain U_1 over U_0, U_1 is properly contained in U_0 and the domain $\omega_0 = U_0 - U_1$ is an annulus of positive modulus. We arbitrarily choose a cross-cut of ω_0, that is, a simple arc which joins a point Q_1 on the inner boundary component of ω_0 to a point Q_0 on the outer boundary component of ω_0 with the property that $P(Q_1) = Q_2$. We let β_0 equal ω_0 with this arc deleted and call the simply connected set β_0 the Carleson box at level zero. Since we assume P is a regular covering of degree 2, this arc pulls back to two disjoint arcs joining the boundary contours of ω_1; these two arcs divide ω_1 into two boxes at level one. Continuing in this manner, $\omega_n = U_n - U_{n-1}$, where $U_n = P^{-n}(U_0)$, is divided into 2^n boxes at level n. Branches of the mapping P^n , for different values of n, give holomorphic homeomorphisms between boxes in the ring ω_{n+k} and boxes in the ring ω_k. As a consequence of the hyperbolic distortion lemma, if we take any two boxes, one from ω_m and one from ω_n for sufficiently large m and n, the homeomorphism of these two boxes will be almost an isometry. Even if m and n are not large, each box is quasi-isometric to a square with unit area and unit side length measured in the Poincaré metric. A bound on the constant of quasi-isometry depends only on the geometry of P. In contrast, the side length of any box in a ring ω_n measured in the Euclidean metric and its distance from the boundary of the unit circle are bounded below and above by positive constants times ε_0^n and ε_1^n, where $0 < \varepsilon_0 < \varepsilon_1 < 1$. This fact follows from Lemma 5, the geometric neighborhood lemma.

The Carleson boxes, so constructed, are quite arbitrary; nonetheless, they lead to definitions and properties which are invariant under the choices made in their construction.

Consider the Banach space B of bounded holomorphic functions φ defined in Δ^* for which

$$||\varphi||_B = \sup_{in\ \Delta^*} |\lambda^{-2}(z)\varphi(z)| < \infty.$$

It is a matter of technical convenience that φ is defined in the whole domain Δ^*; in the end, we only care about its values near the boundary of of the unit circle.

B contains a closed subspace B_0 consisting of those φ such that $|\lambda^{-2}(z)\varphi(z)|$ vanishes at the boundary of the unit circle. More precisely, φ is in B_0 if for every $\varepsilon > 0$, there exists a number $r > 1$, such that

$$\sup_{1<|z|<r} |\lambda^{-2}(z)\varphi(z)| < \varepsilon.$$

For a quadratic-like mapping P, there is an annular neighborhood U_1 inside which it is possible to take the derivative, $\frac{dP}{dz}$, and the derivatives $\frac{dP^n}{dz}$ are defined inside the sets $P^{-n}(U_0)$. Define $P^{n^*}(\varphi)(z)$ to be $\varphi(P^n(z))\left(\frac{dP^n}{dz}(z)\right)^2$ for z in the set $P^{-n}(U_0)$. We can now define quadratic differential forms for P.

Definition. *The space $B(P)$ of bounded holomorphic quadratic differentials for P consists of all functions φ holomorphic in the exterior of the unit circle and such that*

i) φ *is contained in* B *and*

ii) *for every* ε, *there exists an annular neighborhood* $U \subset U_0$, *such that for all* n *and for all* z *in* $P^{-n}(U)$,

$$|P^{n^*}(\varphi)(z) - \varphi(z)| \leq \varepsilon\lambda^2(z).$$

The key element of this definition is that the inequality is uniform in n; it says that if we look in a deep enough annular neighborhood U_k, φ on the ring w_{n+k} is nearly the same as the pullback by P^n of φ on w_k and the amount by which φ and the pullback of φ differ is independent of n.

Proposition. B_0 *is a closed subspace of* $B(P)$ *and* $B(P)$ *is a closed subspace of* B. *Moreover, the natural Banach norm for the quotient space* $B(P)/B_0$ is

$$||\varphi||_{B(P)} = \lim_{U} \sup_{zinU} |\lambda^{-2}(z)\varphi(z)|,$$

where the limit is taken as the annular neighborhood U *shrinks to the boundary of the unit circle.*

Proof. First we show that B_0 is contained in $B(P)$. If φ is in B_0, then for any $\varepsilon > 0$, there exists $r > 1$ so that

$$\sup_{1<|z|<r} |\lambda^{-2}(z)\varphi(z)| < \varepsilon.$$

Thus, for z is in $P^{-n}(U)$, $|\lambda^{-2}(P^n(z))\varphi(P^n(z))| < \varepsilon$. But by the distortion lemma, we can shrink the annular neighborhood U sufficiently so that $\lambda(P^n(z))|P^{n'}(z)| < (1+\varepsilon)\lambda(z)$. We obtain $|\lambda^{-2}(z)(P^{n^*}\varphi)(z)| < \varepsilon(1+\varepsilon)$ and, finally, $|P^{n^*}(\varphi)(z) - \varphi(z)| \leq \{\varepsilon(1+\varepsilon)+\varepsilon\}\lambda^{-2}(z)$, which implies that $\varphi \in B(P)$.

We omit the proof that $B(P)$ is a closed subspace of B since it is so similar to the proof of Theorem 2; the one new ingredient is the hyperbolic distortion lemma (Lemma 3). □

Section 4. An L_1-norm on for $B(P)$ when $P(z) = z^2$

We now construct a second norm for the Banach space of holomorphic quadratic differentials which are automorphic for the dynamical system determined by P. Our definition will depend on the choice of Carleson boxes, but since the norm will turn out to be equivalent to the natural norm for $B(P)/B_0$, this dependence is only apparent.

We assume that $P(z) = z^2$ and that a system of Carleson boxes for P has been constructed. Consider the space $A(P)$ of functions φ holomorphic in Δ^* satisfying

i) $\sup_\beta \int\int_\beta |\varphi| dxdy < \infty$, where the supremum is over all Carleson boxes β, and

ii) for every $\varepsilon > 0$, there exists an annular neighborhood U, such that for all positive integers n and for all z in $P^{-n}(U)$,

$$\sup_\beta \int_\beta \int |P^{n*}(\varphi)(z) - \varphi(z)| dxdy < \varepsilon.$$

Let area (β) denote the Poincaré area of β. We define the $A(P)$ norm of φ to be

$$||\varphi||_{A(P)} = \lim_U \sup_{\beta \text{ in } U} \frac{1}{area(\beta)} \int_\beta \int |\varphi(z)| dxdy,$$

where the supremum is over all Carleson boxes in U and the limit is taken as the annular neighborhood U shrinks towards the boundary. Since the Poincaré areas of all of the Carleson boxes are comparable, the area factor in this definition is superfluous. However, the area factor is necessary if we expect to show that the norm, up to a constant factor, is independent of the choice of Carleson boxes. From the inequality

$$\frac{1}{area(\beta)} \int_\beta \int |\varphi(z)| dxdy \leq \sup_{z \text{ in } \beta} |\varphi(z)\lambda^{-2}(z)| \left(\frac{1}{area\beta} \int_\beta \int \lambda^2(z) dxdy\right),$$

it follows that $||\varphi||_{A(P)} \leq ||\varphi||_{B(P)}$ since the second factor on the right hand side of this inequality is equal to 1. The same type of inequality shows that if φ satisfies condition ii) for $B(P)$ then it satisfies condition ii) for $A(P)$. It follows that restriction of a function φ defined on Δ^* to a function defined on U_0 defines a mapping from $B(P)$ into $A(P)$ which is continuous and that the $A(P)$-norm of any element of B_0 is equal to zero. Provided we show that the automorphy condition ii) for $A(P)$ implies condition ii) for $B(P)$, then the restriction mapping induces a surjection from $B(P)/B_0$ to $A(P)/B_0$ because any function holomorphic in U_0 differs by an element of B_0 from a function which is holomorphic in Δ^* and vanishes of order $|z|^{-4}$ as z approaches ∞. Any Carleson box is surrounded by eight adjacent Carleson boxes. If these boxes are at level n, the Euclidean diameter of each of these eight neighbors is on the order of a constant times $1/2^n$, which is the same as the order of the value of the Poincaré metric in these boxes. From the areal mean value theorem for the subharmonic function $|\varphi(z)|$, we obtain a constant C depending on the geometry of the mapping P such that

$$\sup_{z \ in \ \beta} |\varphi(z)\lambda^{-2}(z)| \leq C \int_{\beta} \int |\varphi(z)| dxdy$$

and it follows that $||\varphi||_{B(P)} \leq C||\varphi||_{A(P)}$. The same argument shows that the automorphy condition ii) for $A(P)$ implies the automorphy condition for $B(P)$.

Theorem 3. *In the case that $P(z) = z^2$ the mapping from $B(P)/B_0$ to $A(P)/B_0$ induced by the restriction of an element of $B(P)$ to $A(P)$ is an isomorphism of Banach spaces.*

Section 5. Lacunary series as quadratic differentials

The preceding theorem gives us equivalent norms for the Banach space $B(P)/B_0$. At this point, we have no examples of nontrivial elements in this Banach space. In this section, we show that $B(P)/B_0$ is infinite dimensional in the case that $P(z) = z^2$.

From $P(z) = z^2$, it follows that $P^k(z) = z^{2^k}$ and $\frac{d}{dz}P^k(z) = 2^k z^{2^k - 1}$. For any holomorphic function Φ, considered as a quadratic differential, define $P^{k^*}\Phi(z)$ to be equal to $\Phi(P^k(z))(\frac{d}{dz}P^k(z))^2$. If we let $\varphi(z)$ be a function holomorphic in Δ^*, by looking at a power series for φ, we find that it is impossible to solve the equation $P^{k^*}\varphi(z) = \varphi(z)$. On the other hand, if we only require equality modulo B_0, this equation has many solutions. They take the form of any linear combination of

$$\varphi_j(z) = \sum_{k=0}^{\infty} \frac{2^{2k}}{z^{2+j2^k}}, \text{ where } j \text{ is an odd integer } \geq 1.$$

The exponents $2 + j2^k$, for $k \geq 0$, coincide with the orbit of $j + 2$ under the mapping $\alpha \mapsto 2\alpha - 2$ acting on the positive integers. Therefore, if we take two unequal odd values of $j \geq 1$, say j_1 and j_2, the exponents of z appearing in the summation for φ_{j_1} will all be distinct from the exponents of z appearing in the summation for φ_{j_2}.

Notice that $\varphi_j(z)$ also can be written in the form

$$\varphi_j(z) = \sum_{k=0}^{\infty} P^{k^*} \Phi_j(z) \ where \ \Phi_j(z) = \frac{1}{z^{j+2}}$$

and so φ_j is a theta-series of the holomorphic function Φ_j whose absolute value is integrable over Δ^*. Thus

$$\varphi_j(z) - P^{n^*} \varphi_j(z) = \sum_{k=0}^{n-1} P^{k^*} \Phi(z)$$

and, therefore, if β is a Carleson box for P in $P^{-n}(U)$,

$$\int_\beta \int |P^{n^*} \varphi_j(z) - \varphi_j(z)| \, dxdy \leq \int_{\beta \cup P(\beta) \cdots \cup P^{n-1}(\beta)} \int |\Phi_j(z)| \, dxdy.$$

Since Φ_j is integrable, this inequality shows that φ_j satisfies the automorphy condition for $B(P)$.

The function φ_j is an element of $B(\Delta^*)$ because its $A(P)$ norm is bounded by a constant times $\int_{\Delta^*} \int |\Phi_j| dxdy$.

A second way to see that φ_j is bounded in $B(\Delta^*)$ is to use the Banach space isomorphism between $B(\Delta^*)$ and the Zygmund bounded vector fields on the unit circle. The isomorphism involves two steps. The first is to integrate φ_j in Δ^* three times to a vector field $V_j(z)\frac{\partial}{\partial z}$. The second step is take the real part of this vector field and restrict its values to the unit circle. For the substitution $z = e^{i\theta}$, restricting z to the unit circle is the same as restricting θ to the real axis. We find that

$$real \ part \ of \ V_j(e^{i\theta})/ie^{i\theta} d\theta = \sum_{k=0}^{\infty} \frac{2^{2k} \, sin(j2^k\theta)}{(j2^k + 1)(j2^k)(j2^k - 1)}.$$

This function is an element of the Zygmund class Λ^* and, on making the substitution $m = j2^k$ and using the identity

$$\frac{m^2}{(m+1)(m)(m-1)} = \frac{1}{m}\{1 + \frac{1}{(m^2-1)}\},$$

we obtain

$$real\ part\ of\ V_j(\theta)/ie^{i\theta}d\theta = \frac{1}{j^2}\sum_{k=0}^{\infty}\frac{1}{j2^k}\{1+\frac{1}{(j^2 2^{2k}-1)}\}\ sin(j2^k\theta).$$

The second term inside the curly bracket approaches zero as k approaches ∞ and, therefore, by the Zygmund-Jackson theorem [8], the contribution of this second term to the summation corresponds to an element of λ^*. Under the isomorphism between Λ^* and $B(\Delta^*)$, this means it corresponds to a cusp form which vanishes at the boundary, namely, to an element of B_0. Since we only care about the class of this function modulo λ^*, which is isomorphic to B_0, we may neglect the second term inside the curly bracket, and the summation takes exactly the form of the classical Weierstrass example, [7, page 48-50]. Our computation shows that the Zygmund norm of V_j is asymptotic to j^{-2}.

Theorem 4. *For the case $P(z) = z^2$, the functions φ_j for j equal to an odd integer bigger than or equal to* 1 *form a linear independent set in the Banach spacez $B(P)/B_0$.*

Proof. We have already shown that the φ_j determine elements of $B(P)/B_0$. We now show that the φ_j's are linearly independent in the quotient Banach space. That is, we show, if a linear combination of the φ_j's is in B_0, then each constant in the linear combination is zero. We actually show much more, namely, that there is a convergent notion of inner product defined for pairs of elements of $B(P)/B_0$ and that, with respect to this inner product, the functions φ_j form a orthogonal family of elements not contained in B_0. This notion of inner product is defined when P is any polynomial-like mapping.

Lemma 4. **(Convergence of inner product)** *Let μ be a P-automorphic Beltrami differential and φ be in $A(P)$. Let w_n be a sequence of ring domains determined by P. Then $\lim_{n\to\infty}(area(w))^{-2}\int\int_{w_n}\mu\varphi\ dxdy$ converges.*

Proof. Because μ and φ are automorphic, we know that for every $\varepsilon > 0$, there exists an annular neighborhood U, such that for all integers $k \geq 0$ and for all z in $P^{-k}(U)$,

$$|P^{k^*}\mu(z)-\mu(z)|<\varepsilon \text{ and } |P^{k^*}\varphi(z)-\varphi(z)|<\varepsilon\lambda^2(z).$$

To show the sequence of the lemma is a Cauchy sequence, we must show that for every $\varepsilon > 0$, there exists an n, such that for all $k > 0$,

$$|\int_{\omega_{n+k}}\int \mu\varphi - 2^k\int_{\omega_n}\int \mu\varphi| = |\int_{\omega_n}\int \mu\varphi - \int_{\omega_{n+k}}\int P^{k^*}\varphi| < (const)\varepsilon 2^{n+k}.$$

Inside the absolute value we subtract and add the term $\int\int P^{k^*}\mu\varphi$, with integration over the domain ω_{n+k}, and then apply the triangle inequality. The resulting two terms are seen to be less than or equal to

$$\varepsilon\int_{\omega_{n+k}}\int|\varphi| + \left(\sup_{z \text{ in } \omega_{n+k}}|P^{k^*}\mu(z)|\right)\varepsilon\int_{\omega_{n+k}}\int\lambda^2.$$

The first term is less than a constant times $\varepsilon\ 2^{n+k}$ because φ is in $A(P)$. The second term is bounded similarly because μ is bounded and because of the hyperbolic distortion lemma. □

From the hyperbolic distortion lemma, one can show that for ψ in $A(P)$, $\lambda^{-2}\overline{\psi}$ is P-automorphic Beltrami differential, (see Lemma 3 of section 2). From the above lemma we conclude that the limit in the following inner product converges:

$$<\varphi,\psi>=\lim_{n\to\infty}\frac{1}{area(\omega_n)}\int_{\omega_n}\int\varphi(z)\lambda^{-2}(z)\overline{\psi(z)}\,dxdy$$

.

It is easy to see that $<\varphi,\varphi>^{1/2}\leq ||\varphi||_{B(P)/B_0}$. The ring domains for $P(z) = z^2$ can be taken to be bounded by concentric circles. If we take two P-automorphic forms φ_{j_1} and φ_{j_2} with j_1 not equal to j_2 we obtain $<\varphi_{j_1},\varphi_{j_2}>=0$ since none of the exponents of z occurring in the series for φ_{j_1} coincides with any exponent of z occurring in the series for φ_{j_2}.

We now show that $<\varphi_j,\varphi_j>$ is asymptotic to a positive constant times j^{-4} and this will complete the proof of the theorem. We take ω_n to be the region between $(1+\varepsilon)$ and $(1+\varepsilon)^2$. Then the Poincaré area of ω_n is asymptotic to $2\pi/\varepsilon$ and

$$< \varphi_{j'}\varphi_j = \lim_{\varepsilon\to 0} \varepsilon \int_{1+\varepsilon}^{1+\varepsilon)^2} \sum_{k=o}^{\infty} \frac{2^{4k}(r\ log\ r)^2}{r^{4+j2^{k+1}}}\ rdr = \lim_{\varepsilon\to 0} 4\ \varepsilon^4 \sum_{k=0}^{\infty} \frac{2^4 k}{(1+\varepsilon)^{j2^k}}.$$

Since the omission of any finite number of terms in this summation does not affect the limit as e approaches zero, we see that the inner product norm of φ_j depends only on the tail of the infinite series of its definition. To estimate this limit, we replace the summation by an integral and integrate by parts three times. We obtain $< \varphi_{j'}, \varphi_j >= (constant)\ j^{-4}$ and so the Zygmund norm of V_j and the inner product norm of φ_j are both asymptotic to j^{-2}. □

Section 6. The Bers' $\mathcal{L}$-operator

The Bers' $\mathcal{L}$-operator [1] is a device which converts Beltrami differentials into holomorphic quadratic differentials. The $\mathcal{L}$-operator arises from analyzing the derivative of composition of mappings representing points in Teichmüller space. More precisely, for the quasisymmetric mappings f and h, if we hold h fixed and let f vary, the mapping $[f] \mapsto [f \circ h]$ is differentiable. Calculating the expression for its derivative, followed through the conformal welding process, gives rise to the $\mathcal{L}$-operator.

In the next section, we use the $\mathcal{L}$-operator and the existence of quadratic differentials for the special polynomial mapping $P(z) = z^d$ which was shown in the previous section to deduce the existence of quadratic differentials for arbitrary polynomial-like mappings of degree d. In this section we show that Bers' $\mathcal{L}$-operator preserves P-automorphic forms.

It is convenient to transport our spaces of differentials to spaces of differential forms on the logarithmic covering space. The lifting of a degree d mapping to this covering is an injective mapping. This convenient fact was already used in section 2 to prove the hyperbolic distortion lemma. As before, for simplicity of notation, we assume that $d = 2$, but all the theorems apply for arbitrary positive integers $d \geq 2$.

In our set-up, we assume we have a quasiconformal conjugation of the lifting of a quadratic-like mapping. The picture is of a quasicircle C, which is periodic in the sense that $C + 2\pi i = C$. (Think of C as the imaginary axis or a periodic quasiconformal distortion of the imaginary axis.) The complement of C consists of two simply connected domains, L and R, which we refer to as the left side and the right side. There is a one-to-one conformal mapping α, the lift of the quadratic-like mapping P, which is defined in a neighborhood of C, periodic in the sense that $\alpha(z+2\pi i) = \alpha(z) + 4\pi i$, leaves C invariant, and is expanding in the following sense. There are periodic open sets U_0 and U_1 contained in L and V_0 and V_1 contained in R such that $U_1 \cup C \cup V_1$ is an open neighborhood of C in the complex plane. The mapping α fixes the set C and maps U_1 onto U_0 and V_1 onto V_0. The topological periodic strip domains $U_0 - U_1$ and $V_0 - V_1$ factored by the translation $z \mapsto z + 2\pi i$ are conformal cylinders with positive moduli. By following the backwards images of U_0 under iterates of α, we obtain a sequence of strips $U_0 \supset U_1 \supset \cdots \supset U_n \supset \cdots$ and the modulus of each cylinder $(U_{n-1} - U_n)/(z \mapsto z + 2\pi i)$ has twice the modulus of the preceding cylinder $(U_n - U_{n+1})/(z \mapsto z + 2\pi i)$. We have a similar picture on the right hand side R where there is a sequence of strip domains $V_0 \supset V_1 \supset \cdots \supset V_n \cdots$, which are the backwards images of V_0 under the iterates of α. It is convenient to let $W_n = U_n \cup C \cup V_n$, so α^n maps W_n injectively and holomorphically onto W_0. The log of the absolute value of the derivative of α is bounded above and below by positive constants throughout the closure of W_1.

We need a lemma concerning strip neighborhood systems. For any z, let $\delta(z)$ be the minimum Euclidean distance from z to C. Consider the sequence of neighborhoods of C given by $U_n(\varepsilon) = \{z : z \in R \text{ and } \delta(z) < \varepsilon^n\}$ We call such a sequence $U_n(\varepsilon)$ a geometric system of neighborhoods.

Lemma 5. (**Geometric neighborhoods**) *Given any strip neighborhood U contained in R, there exists geometric systems of neighborhoods $U_n(\varepsilon_0)$ and $U_n(\varepsilon_1)$ with $0 < \varepsilon_0 < \varepsilon_1 < 1$ and an integer k such that $U_{n+k}(\varepsilon_0) \subseteq \alpha^{-n}(U) \subseteq U_{n-k}(\varepsilon_1)$ for all $n \geq k$. The numbers ε_0 and ε_1 depend on the largest and smallest values of $|\alpha'|$ in W_1.*

Proof. The lemma is obviously true in the case that $P_0(z) = z^2$ with $\varepsilon_0 = \varepsilon_1 = 1/2$. Any P is related to P_0 by the equation $h \circ P_0 \circ h^{-1} = P$, where h is quasiconformal and h preserves the unit circle. The lemma follows from the observation that h is quasiconformal and, hence, both h and h^{-1} are Hölder continuous. □

In the right hand side R, we consider the Banach space $L_\infty(\alpha)$ of Beltrami differentials for α. It consists of L_∞ -complex-valued functions μ defined on R satisfying

i) $\mu(z+2\pi i)=\mu(z)$ and

ii) for every $\varepsilon>0$, there exists a periodic strip neighborhood V of C in R, such that for all $n\geq 0$ and for all z in $\alpha^{-n}(V)$,

$$|\mu(\alpha^n(z))\frac{\overline{\alpha^{n'}(z)}}{\alpha^{n'}(z)}-\mu(z)|<\varepsilon.$$

On the left side L, we consider the Banach space $B(\alpha)$ of holomorphic functions φ which are bounded cusp forms in the sense that, if λ is the Poincaré metric for L, then $\sup_{z \text{ in } L}|\varphi(z)\lambda^{-2}(z)|<\infty$,

i) $\varphi(z+2\pi i)=\varphi(z)$ and

ii) for every $\varepsilon>0$, there exists a periodic strip neighborhood U of C in L, such that for all $n\geq 0$ and for all z in $\alpha^{-n}(U)$,

$$|\varphi(\alpha^n(z))\alpha^{n'}(z)^2-\varphi(z)|<\varepsilon\lambda^2(z).$$

The next lemma enables us to view condition ii) in an apparently weaker form.

Lemma 6. **(Bootstrapping)** *Suppose ψ is a function holomorphic in L such that $\psi(z+2\pi i)=\psi(z)$ and the* sup $|\lambda^{-2}(z)\psi(z)|$ *over z in L is bounded. Suppose further that there exists a positive integer j, such that for every $\varepsilon>0$, there exists a strip neighborhood U, such that for all positive integers n and for all z in $\alpha^{-nj}(U)$, $|\psi(\alpha^n(z))\alpha^{n'}(z)^{-2}-\psi(z)|<\varepsilon\lambda^2(z)$. Then ψ is in $B(\alpha)$.*

Proof. We apply the hypothesis j times. For z in $\alpha^{-nj}(U)$,

$$|\alpha^{jn^*}\psi-\alpha^{j(n-1)^*}\psi|\leq\varepsilon\alpha^{n(j-1)^*}\lambda^2,$$

$$|\alpha^{j(n-1)^*}\psi-\alpha^{j(n-2)^*}\psi|\leq\varepsilon\ \alpha^{n(j-2)^*}\lambda^2,\ldots$$

$$|\alpha^{j^*}\psi-\psi|\leq\varepsilon\ \lambda^2.$$

Summing these inequalities and using the hyperbolic distortion lemma, we find that for every z in $\alpha^{-jn}(U)$, $|\alpha^{jn^*}\psi(z)-\psi(z)|1\leq(const)j\varepsilon\lambda^2(z)$. A simple modification of this argument applies to arbitrary integers of the form $jn+m$, where m is between 0 and $j-1$. $\square$

Lemma 7. **(Schwarzian distortion)** *There is a constant* ε_0 *with* $0<\varepsilon_0\leq 1/2$ *depending on* α*, such that for every* ε *with* $0<\varepsilon<\varepsilon_0$ *and every* z *and* ζ in W_n *with* $|z-\zeta|<\varepsilon\varepsilon_0^n$,

$$A)\quad \left|\frac{\alpha^{n'}(z)\alpha^{n'}(\zeta)}{(\alpha^n(z)-\alpha^n(\zeta))^2}-\frac{1}{(z-\zeta)^2}\right|<\frac{\varepsilon}{|z-\zeta|^2}.$$

Moreover,

$$B)\quad \left|\frac{\alpha^{n'}(z)^2\alpha^{n'}(\zeta)^2}{(\alpha^n(z)-\alpha^n(\zeta))^4}-\frac{1}{(z-\zeta)^4}\right|<\frac{\varepsilon}{|z-\zeta|^4}.$$

Proof. The Schwarzian derivative Sf of a holomorphic function f is equal to

$$Sf(z)=6\lim_{\zeta\to z}\frac{\partial^2}{\partial z\partial\zeta}log\frac{f(z)-f(\zeta)}{z-\zeta}=6\lim_{\zeta\to z}\left(\frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2}-\frac{1}{(z-\zeta)^2}\right).$$

We define the bi-Schwarzian derivative S to be the same expression without the constant 6 and without taking the limit as ζ approaches z:

$$Sf(z,\zeta)=\frac{\partial^2}{\partial z\partial\zeta}log\frac{f(z)-f(\zeta)}{z-\zeta}.$$

The composition law for the bi-Schwarzian is

$$S(f\circ g)(z,\zeta)=(Sf)(g(z),g(\zeta)g'(z)g'(\zeta)+Sg(z,\zeta).$$

Since α is assumed to be univalent in an open neighborhood of W_1, $S\alpha(z)$ is bounded on W_1 and the bi-Schwarzian $S\alpha(z,\zeta)$ is bounded on W_1XW_1. On taking the bi-Schwarzian of α^n, we obtain

$$(S\alpha)(\alpha^{n-1}(z),\alpha^{n-1}(\zeta))\alpha^{n-1'}(z)\alpha^{n-1'}(\zeta)$$

$$+(S\alpha)(\alpha^{n-2}(z),\alpha^{n-2}(\zeta))\alpha^{n-2'}(z)\alpha^{n-2'}(\zeta)+\cdots+S\alpha(z,\zeta)$$

and, on letting K_1 be a bound for $S\alpha$ on W_1XW_1 and k_1 a bound for α' on W_1, we see that if z and ζ are in W_n this expression is bounded by

$$K_1\{1+k_1^2+\cdots+(k_1^2)^{n-1}\} \le K_1\frac{k_1^{2n}-1}{k_1^2-1} \le (const)k_1^{2n}.$$

Thus, if $|z-\zeta| < \varepsilon\varepsilon_0^n$, then $|z-\zeta|^2$ multiplied by the left hand side of A) is less than $(const)\varepsilon^2\varepsilon_0^{2n}k_1^{2n}$. Part A) follows by choosing ε_0 small enough so that $\varepsilon_0 k_0 \le 1$ and $(const)\varepsilon_0 \le 1$.

To prove part B), we apply the factorization $C^2 - D^2 = (C-D)(C+D)$ and the result of part A). On multiplying both sides of the inequality in B) by $|z-\zeta|^4$, the result follows from part A and the bound

$$\left|\frac{\alpha^{n'}(z)\alpha^{n'}(\zeta)(z-\zeta)^2}{(\alpha^n(z)-\alpha^n(\zeta))^2}\right| \le k_2^{2n}$$

where k_2 is chosen so that $|\alpha'(p)/\alpha'(q)| < k_2$ for every p and q in W_1. Of course, the maximum value of $|\alpha''/\alpha'|$ times the Euclidean diameter of $W_1\cap$(the horizontal strip between $y=0$ and $y=2\pi$) is a bound for $\log k_2$. To guarantee the inequality in part B, we choose ε_0 so that $\varepsilon_0 k_1 k_2 \le 1$ and $(const)\varepsilon_0 \le 1$. This completes the proof of the Schwarzian distortion lemma. □

The $\mathcal{L}$-operator is defined by the formula

$$\mathcal{L}\mu(z)dz^2 = \psi(z)dz^2 = \int_{V_0}\int \frac{\mu(\zeta)d\zeta d\bar{\zeta}}{(\zeta-z)^4}dz^2.$$

In this definition we could take the domain of integration to be the lift of any annular neighborhood of the circumference of the unit circle, since, in the end, we only care about the equivalence class of ψ modulo B_0.

Theorem 5 *The $\mathcal{L}$-operator is a bounded linear mapping from the Banach space $L_\infty(\alpha)$ to the Banach space $B(\alpha)/B_0$.*

Proof. In the classical case, the domain of integration is the whole right hand side, R, and α is a Möbius transformation preserving R and the conditions ii) in the definitions of $L_\infty(\alpha)$ and $B(\alpha)$ are replaced by exact automorphy, (the inequalities are true for $\varepsilon = 0$). Then this Theorem follows by changing variables and using the identity $(\alpha(z)-\alpha(\zeta))^2 = \alpha'(z)\alpha'(\zeta)(z-\zeta)^2$, which is true if α is a Möbius transformation. In the situation at hand, we only know that the holomorphic mapping α arises from an expanding polynomial-like mapping and, hence, we may not use this identity.

Let $K(z,\zeta) = (z-\zeta)^{-4}, \alpha^* K(z,\zeta) = \alpha'(z)^2\alpha'(z)^2\Big(\alpha(z)-\alpha(\zeta)\Big)^4$ and $\alpha^*\mu(z) = \mu(\alpha(z)\frac{\overline{\alpha}'(z)}{\alpha'(z)}$. To prove the theorem we must estimate

$$\mathcal{L}\varphi(\alpha^n(z))\alpha^{n'}(z)^2 - \mathcal{L}\varphi(z).$$

We write this difference as a sum of three terms:

$$I(z) = \int_{V_n}\int (\alpha^{n^*}\mu)(\zeta)(\alpha^{n^*}K(z,\zeta) - K(z,\zeta))d\xi d\eta,$$

$$II(z) = \int_{V_n}\int \Big((\alpha^{n^*}\mu)(\zeta) - \mu(\zeta)\Big) K(z,\zeta)d\xi d\eta, \ \textit{and}$$

$$III(z) = \int_{V_0 - V_n}\int \mu(\zeta)K(z,\zeta)d\xi d\eta.$$

Each of these integrals is a periodic function of z and holomorphic for values of z in L. To prove ii) in the definition of $B(\alpha)$ we repeatedly use the fact that

$$\int_{|\zeta - z|>\gamma}\int |\zeta - z|^{-4} d\xi d\eta = \pi\gamma^{-2}.$$

From the bootstrapping lemma, it suffices to find j such that for all $\varepsilon > 0$, there exists a k, such that for all $n \geq 0$ and for all z in $\alpha^{-jn}(U_k) = U_{jn+k}$, $|\alpha^{n^*}\psi(z) - \psi(z)| < \varepsilon\lambda(z)^2$. From the geometric neighborhood lemma, select β_1 and k so that $U_m \subseteq U_{m-k}(\beta_1)$. If z is in $U_{jn+k'}$, then $\delta(z) < \beta_1^{jn}$. Select j so that $\beta_1^j < \varepsilon_0$, where ε_0 is the value in the Schwarzian distortion lemma. To estimate the integral $I(z)$, take a disk of radius ε_0^n centered at z and write the integral as a sum of two integrals, the first over the $V_n\cap$(the disk) and the second over $V_n\cap$(the complement of the disk). By the Schwarzian distortion lemma, the first integral is bounded by

$$\varepsilon||\mu||_\infty 2\pi\int_{\delta(z)}^{\infty}\frac{dr}{r^3} \leq \varepsilon\pi||\mu||_\infty\delta(z)^{-2}.$$

The second integral is bounded by $\pi||\mu||_\infty(\varepsilon_0^{-2n} + \varepsilon_0^{-2n}k_2{}^{2n})$. Now, put the additional condition on j that $\beta_1^j < \varepsilon_0/k_2$ and we obtain the desired estimate.

To estimate $II(z)$, the assumption ii) in the definition of μ implies, for every $\varepsilon > 0$, there exists an n_0, such that for all $n \geq 0$ and for all z with $\delta(z) < \varepsilon_0^{n+n_0}$, $|P^{n^*}\mu(z) - \mu(z)| < \varepsilon$. Consider a disk centered at a point z in L and of radius $\varepsilon_0^{n+n_0}$. The integral $II(z)$ is bounded by the sum of an integral over $V_n \cup$(this disk) plus an integral over $V_n \cup$(the complement of this disk). The first integral is bounded by a constant times

$$\varepsilon \int_{\delta(z)}^{\varepsilon_0^{n+n_0}} \frac{dr}{r^3} \leq \varepsilon \ (const) \ \delta(z)^{-2}.$$

The second integral is bounded by $2\|\mu\|_\infty(1/\varepsilon_0^{2n+2n_0})$. Pick β_1 so that $\alpha^{-n}(U_0) \subseteq U_{n-k}(\beta_1)$. If j is so large that $\beta_1^j < \varepsilon_0$ and if $\delta(z) < \beta_1^{nj}$, then this term grows much more slowly than $\delta(z)^{-2}$ and we again obtain the desired estimate.

To estimate $III(z)$, select systems of geometric neighborhoods $V_n(\varepsilon_0)$ and $U_n(\beta_1)$ and a positive integer k, such that $V_{n+k}(\varepsilon_0) \subseteq V_n$ and $U_n \subseteq U_{n-k}(\beta_1)$. Then $III(z)$ is bounded by $\pi\|\mu\|_\infty/(\ dist(z,\ right\ hand\ bdry\ of\ V_n))^2$ which is less than $\pi\|\mu\|_\infty/(\delta(z)+\varepsilon_0^{n+k})^2 = \pi\|\mu\|_\infty/\delta(z)^2(1+\varepsilon_0^{n+k}/\delta(z))^2$. If we pick z in U_{jn+k}, then z is in $U_{jn}(\beta_1)$ then $\varepsilon_0^{n+k}/\delta(z) > \varepsilon_0^{n+k}/\beta_1^{jn}$ and, by choosing j large enough, this fraction is as large as we like. We obtain the desired inequality and this concludes the proof of the theorem. □

Theorem 6. *The Bers' $\mathcal{L}$-operator induces an isomorphism from $B(\alpha, R)/B_0$ onto $B(\alpha, L)/B_0$, where B_0 is the Banach space of periodic bounded cusp forms which vanish at C.*

Proof of theorem. We define a modified $\mathcal{L}$-operator, $\hat{\mathcal{L}}$ by

$$\hat{\mathcal{L}}\psi(z) = \int_{V_0}\int \frac{\lambda^{-2}(\zeta)\overline{\psi(\zeta)}d\xi d\eta}{(\zeta - z)^4}.$$

We have already observed that the hyperbolic distortion lemma implies that if ψ is an element of $B(\alpha, R)$ then $\lambda^{-2}\overline{\psi}$ is an element of $L_\infty(\alpha, R)$. The previous theorem then implies that $\hat{\mathcal{L}}\psi$ is an element of $B(\alpha, L)$. It is obvious that $\hat{\mathcal{L}}$ preserves the periodic bounded cusp forms which vanish at C. It is shown by Bers in [1] that $\mathcal{L}$ from $L_\infty(R)$ to $B(\alpha, L)$ and that $\hat{\mathcal{L}}$ from $B(R)$ to $B(L)$ is an injection. Thus $\hat{\mathcal{L}}$ restricted to $B(\alpha, R)$ is injective. Bers' method of proof relies on a reproducing formula and an antiquasiconformal involution χ which fixes the curve C pointwise, which is periodic and which commutes have a uniform lower bound on the Teichmüller radius of the open sets on which these charts are defined. □

Recently, Jeremy Kahn has shown that any of the coordinate chart mappings described in this theorem are globally one-to-one. They therefore embed T as a domain in $B(P)/B_0$. The argument is elementary and applies to the QS/S Teichmüller space described in [4].

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